Algebraic Geometry

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Contents

These are lecture notes for a course about algebraic geometry taught in the winter term 2021/22 in Münster. The goal of the course was to cover the basics of algebraic geometry: schemes, sheaves, $\mathcal{O}_X$-modules. The preliminaries for the course are a good knowledge of rings, fields, ideals, prime ideals and very importantly the basic concepts of categories. Moreover a knowledge of the basics of topological spaces would be helpful.
Literature that I have used to prepare the course is:

(1) R. Hartshorne: Algebraic Geometry GTM 52. Springer.
(2) D. Mumford: The red book of varieties and schemes. Springer LN 1358.
(3) U. Goertz, T. Wedhorn: Algebraic Geometry I. Vieweg.
(4) A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique.
(5) P. Scholze: Algebraic Geometry I lecture notes (typed by Jack Davies)

CHAPTER 1

Algebraic Geometry I - Wintersemester 2021/22

1. Affine Varieties

What is the idea of algebraic geometry? We first look at some analogies:

- Linear algebra $\simeq$ study systems of linear equations. Solutions form a sub-vectorspace

- Algebra $\simeq$ study a single polynomial equation, i.e. zeros of polynomials in $\mathbb{C}$ with rational coefficients (e.g. $\mathbb{Q}(\zeta_n)$)

- Algebraic geometry $\simeq$ study solutions to several polynomial equations, e.g $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid f_1(x) = ... = f_k(x) = 0\}$ with $f_1, .., f_k \in \mathbb{R}[X_1, ..., X_n]$.

Think geometrically, i.e. consider $\{x^2 + y^2 = 1\}$ as a circle, not just as a set of elements in $\mathbb{R}^2 \Rightarrow$ transfer geometric intuition to solve algebraic questions and vice versa.

Remark 1.1. Recall that a smooth manifold M of dimension d can be defined as a subset $M \subseteq \mathbb{R}^{d+k}$ for which there are (locally) smooth functions $f_1, ..., f_k : \mathbb{R}^{d+k} \to \mathbb{R}$ such that

$$M = \{x \in \mathbb{R}^{d+k} \mid f_1(x) = f_2(x) = ... = f_n(x) = 0\}$$

and such that for each $x \in M \subseteq \mathbb{R}^{d+k}$ the Jacobi matrix $(\partial f_i(x)/\partial x_j)_{i,j}$ has maximal rank.

The main difference that we make is to restrict to polynomial equations as opposed to smooth funtions (we are doing algebra after all) and to drop the rank condition which is there to make sure that the subset is smooth (we will get back to smoothness in the algebraic setting later). We also remark that a set of solutions to smooth equations $M \subseteq \mathbb{R}^n$ is automatically closed and vice versa that each closed subset can be described by smooth functions if we drop the rank condition (by a theorem of Whitney even by a single function).
Now we let k be a field (we will soon assume that k is algebraically closed).

Definition 1.2. *An affine algebraic set is a subset $V \subseteq k^n$ that can be written as the solutions of a set of polynomial equations, i.e. there is a set $M \subseteq k[X_1, ..., X_n]$ such that*

$$V = V(M) := \{(x_1, ..., x_n) \mid f(x_1, .., x_n) = 0 \textit{ for all } f \in M\}$$

Example 1.3. (1) The equation $y = ax + b$ gives us a straight line. Similarly all equations of degree 1 in two variables give straight lines.
(2) The "circle", i.e. the set of elements $(x, y) \in k^2$ such that $x^2 + y^2 = 1$.
(3) We similarly have $x^2 + y^2 = -1$, which has no real solutions

(4) The equation $x^2 = y^2$ can be factored as $(x + y)(x - y) = 0$ so that it describes a union of two diagonal lines through the origin. The origin is a "singularity", at least in the smooth world, since the derivative of the polynomial vanishes at the origin.
(5) The elliptic curve $y^2 = x^3 - x + 1$ which is smooth (at least over $\mathbb{R}$ as a smooth manifold)
(6) Another elliptic curve is $y^2 = x^3 - x$ which is also smooth but disconnected.
(7) The elliptic curve $y^2 = x^3 - x^2$ has a "nodal singularity" at the origin.
(8) The curve $y^2 = x^3$ has a "cuspidal singularity" at the origin.

Recall that for every subset $M \subseteq R$ of a commutative ring R there is a smallest ideal containing M. Elements of this ideal will be linear combinations $\sum \lambda_i m_i$ for $m_i \in M$ and $\lambda_i \in R$. If the set $M = \{r_1, ..., r_n\}$ is finite we write this ideal as $(r_1, ..., r_n)$.

PROPOSITION 1.4. *For any subset $M \subseteq k[X_1, ..., X_n]$ let $\mathfrak{a}$ be the ideal generated by M. Then we have that*

$$V(M) = V(\mathfrak{a}) \ .$$

PROOF. Since $M \subseteq \mathfrak{a}$ we clearly have $V(\mathfrak{a}) \subseteq V(M)$. On the other hand, if $x \in V(M)$ and $g = \sum \lambda_i m_i \in \mathfrak{a}$ then

$$g(x) = \sum \lambda_i(x) m_i(x) = 0 \ . \qquad \square$$

Now our first big goal is to prove the following result:

THEOREM 1.5. *Every affine algebraic set $V \subseteq k^n$ is the solution set of finitely many polynomials $f_1, ..., f_k$.*

In order to prove this, it is enough to prove that for a given subset $M \subseteq k[X_1, ..., X_n]$ we find finitely many polynomials $f_1, ..., f_k$ such that

$$V(M) = V(f_1, ..., f_k) \ .$$

We already know that $V(M)$ equals $V(\mathfrak{a})$. Thus to prove the theorem is suffices to prove that every ideal $\mathfrak{a} \subseteq k[X_1, ..., X_n]$ is finitely generated, i.e. of the form

$$\mathfrak{a} = (f_1, ..., f_k) \ .$$

We invent a name for such rings.

2. Noetherian rings

DEFINITION 2.1. *A (commutative)*[1] *ring R is called* noetherian *if all ideals are finitely generated, i.e. if for each Ideal $\mathfrak{a} \subseteq R$ there are finitely many element $r_1, ..., r_k$ such that*

$$\mathfrak{a} = (r_1, ..., r_k) \ .$$

EXAMPLE 2.2. Fields are noetherian. Finite rings are noetherian (we simply can take all elements of the ideal as generators). The integers are noetherian, as every ideal is a principal ideal (in some sense noetherian rings are generalization of principal ideal rings).

PROPOSITION 2.3. *The following statements are equivalent for a ring R:*

(1) *R is noetherian.*

[1] Henceforth all rings will be commutative with 1

(2) *Each ascending chain of ideals*

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq ...$$

becomes eventually constant

(3) *Each non-empty subset of ideals has a maximal element.*

PROOF. Assume (1) and

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq ...$$

a chain of ideals. Then we consider the union $\bigcup \mathfrak{a}_i \subseteq R$. This is an ideal. Thus it has to be finitely generated. But each generator lies in some finite step, so that the ideal equals already a finite union.
(2) $\Rightarrow$ (3): Assume (3) fails for some set S of ideals. Then we find for each ideal $I \in S$ an ideal $J \in S$ with $I \subsetneq J$ (as otherwise I would be maximal). Inductively, this allows us to build a sequence of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq ...$ in contradiction to (2).
Assume (3) and let $I \subseteq R$ be an ideal. Consider the set of finitely generated ideals contained in I. Then this set is non-empty and has a maximal element $I' \subseteq I$. But for any $a \in I$ we find that $I' + (a)$ is also a finitely generated ideal contained in I, so by maximality of I' we have that $I' + (a) = I'$ for each $a \in I$. It follows that each such a lies already in I', i.e. $I = I'$ and I is finitely generated. $\square$

EXAMPLE 2.4. Consider the ring $R = k[X_i \mid i \in \mathbb{N}]$ which is given by polynomials in countably many variables. Formally this ring is the colimit of the finite polynomial rings, concretely every polynmial has only finitely many variables. We claim that R is not noetherian. To see this consider the chain

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq$$

LEMMA 2.5. *Quotients of noetherian rings are noetherian.*

PROOF. Let $J \subseteq R/I$ be an ideal. Then the preimage $J' \subseteq R$ of all elements in R that map to J is by assumption finitely generated. It follows that also J is finitely generated. $\square$

EXAMPLE 2.6. Subrings of noetherian rings need not be noetherian. To see this consider the non-noetherian ring $R = \mathbb{Q}[X_i|i \in \mathbb{N}]$. This ring does not have zero-divisors, so it is a subring of its fraction field $Q(R)$, which is noetherian.

THEOREM 2.7 (Hilbert's basis theorem). *Assume R is noetherian, then so is $R[X]$.*

PROOF. Let $I \subseteq R[X]$ be an ideal which is not finitely generated. We construct inductively a sequence of polynomials

$$f_0 = 0, f_1, f_2, ...$$

by letting f_i be a polynomial of minimal degree in $I \setminus (f_1, ..., f_{i-1})$. Let d_n be the degree of f_n and a_n be the leading coefficient. Note that the sequence of d_n is nondecreasing by construction, since f_n is of minimal degree in $I \setminus (f_1, \dots, f_{n-1}) \ni f_{n+1}$.
Then consider the chain of ideals in R:

$$(a_1) \subseteq (a_1, a_2) \subseteq (a_1, a_2, a_3) \subseteq ...$$

Since R is noetherian, this sequence becomes constant, say at stage n. Then

$$a_{n+1} \in (a_1, ..., a_n) \ .$$

so that $a_{n+1} = \lambda_1 a_1 + ... + \lambda_n a_n$ and we can consider the polynomial

$$f_{n+1} - \sum_{i=1}^{n} \lambda_i X^{d_{n+1}-d_i} f_i$$

This is in $I \setminus (f_0, ..., f_n)$ since the second term is in $(f_0, \dots, f_n)$. But on the other hand its degree is less than the degree of f_{n+1} in contradiction to the choice of f_{n+1}. □

Corollary 2.8. *For every noetherian ring R we have that $R[X_1, ..., X_n]$ is noetherian.*

3. The Zariski topology

Okay, lets get back to affine varieties. Recall that we now have proven Theorem 1.5. We know that affine algebraic sets $V \subseteq k^n$ are the zero sets of finitely many polynomials. Let us recall that a topology on a set X corresponds on a set of subsets of X called *open subsets* such that

(1) $\emptyset$, X are open.
(2) Arbitrary unions of open sets are open.
(3) Finite intersections of open sets are open.

A subset $K \subset X$ is then called closed, if the complement $X \setminus K$ is open. One immediately sees that

(1) $\emptyset$, X are closed.
(2) Arbitrary intersections of closed sets are closed.
(3) Finite unions of closed sets are closed.

One could clearly have defined a topological space equivalently by means of its closed subsets.

Proposition 3.1. *There is a topology on k^n for which the closed subsets are the affine algebraic sets.*

Proof. We have $\emptyset = V(1)$ and $k^n = V(\emptyset)$. Moreover we have that

$$\bigcap_{i \in I} V(M_i) = V(\bigcup_{i \in I} M_i)$$

so that the intersection of closed sets is closed. Finally we have to prove that the union $A \cup B$ of two closed sets

$$A = V(M) \qquad \text{and} \qquad B = V(N)$$

is again closed. We consider

$$M \cdot N = \{f \cdot g \mid f \in M, g \in N\}$$

and claim that $A \cup B = V(M \cdot N)$. The inclusion $\subseteq$ is clear. For the converse assume that $(x_1, ..., x_n)$ is a zero of all polynomials $f \cdot g$ and not in A, i.e. there is at least one $f_0 \in M$ with $f(x_1, ..., x_n) \neq 0$. For each $g \in N$ we get that $f_0(x_1, ..., x_n) \cdot g(x_1, ..., x_n) = 0$ so that $g(x_1, ..., x_n) = 0$ and thus $(x_1, ..., x_n) \in B$. □

Definition 3.2. *The topology is called the* Zariski topology*. We denote the space k^n with the Zariski topology by $\mathbb{A}^n(k)$ or simply $\mathbb{A}^n$ if the ground-field is clear. It is called the n-dimensional affine space.*

EXAMPLE 3.3. Identity $\mathrm{Mat}(n \times n, k) = k^{n^2} = \mathbb{A}^{n^2}$. Then the subset

$$\mathrm{Gl}(n,k) \subseteq \mathrm{Mat}(n \times n, k)$$

of invertible matrices is Zariski open. To see this we note that the complement, i.e. the set of matrices with vanishing determinant, is closed. But this is true by definition since it is $V(\det)$ where $\det : \mathrm{Mat}(n \times n, k) \to k$ is considered as a polynomial in n^2 entries.
We can also consider $\mathrm{Gl}(n,k)$ as an affine algebraic set as follows: consider the identification

$$\mathrm{Gl}(n,k) \cong \{(A,y) \in \mathrm{Mat}(n \times n, k) \times k \mid \det(A) \cdot y = 1\} \subseteq k^{n^2} \times k = \mathbb{A}^{n^2+1}$$

in which case it is the vanishing locus of a single polynomial in n^2+1 entries, namely $\det \cdot Y - 1 \in k[X_1, ..., X_{n^2}, Y]$. We will consider the latter the 'standard way' of viewing it as an affine algebraic set.

Now we want to investige how much the algebraic set $V = V(\mathfrak{a})$ remembers about the ideal $\mathfrak{a}$.

DEFINITION 3.4. *Let $\mathfrak{a} \subseteq R$ be an ideal. The* radical *of $\mathfrak{a}$ is the ideal*

$$\sqrt{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$$

An ideal is called radical (or radically closed) if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

LEMMA 3.5. *$\sqrt{\mathfrak{a}}$ is itself an ideal and we have $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.*

PROOF. Clearly for $x \in \sqrt{\mathfrak{a}}$ and $r \in R$ we have that $(rx)^n = r^n x^n \in \mathfrak{a}$. Thus $rx \in \sqrt{\mathfrak{a}}$.
For $x, y \in \sqrt{\mathfrak{a}}$ with $x^n, y^n \in \mathfrak{a}$ we find that $(x+y)^{2n} \in \mathfrak{a}$ since each monomial $x^i y^j$ with $i + j = 2n$ has to contain at least n factors of x or n factors of y.
The other claim is clear. $\square$

LEMMA 3.6. *For an ideal $\mathfrak{a} \subseteq k[X_1, ..., X_n]$ we have that*

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$$

PROOF. The containment $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{a})$ is obvious since $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$. For the converse assume that $(x_1, ..., x_n) \in V(\mathfrak{a})$ and $p \in \sqrt{\mathfrak{a}}$. Then there exists an n such that $p^n \in \mathfrak{a}$, in particular $p^n(x_1, ..., x_n) = 0$. But then also $p(x_1, ..., x_n) = 0$. Thus $(x_1, ..., x_n) \in V(\sqrt{\mathfrak{a}})$. $\square$

For any subset $M \subseteq \mathbb{A}^n$ we define the *vanishing ideal* $I(M) \subseteq k[X_1, .., X_n]$ as the set of elements

$$I(M) = \{f \in k[X_1, ..., X_n] \mid f(x_1, ..., x_n) = 0 \text{ for all } x \in M\} .$$

This is clearly a radical ideal.

THEOREM 3.7 (Hilbert's Nullstellensatz). *Assume that k is algebraically closed. Then the map*

$$V(-) : \quad \{\textit{radical ideals } \mathfrak{a} \subseteq k[X_1, ..., X_n]\} \to \{\textit{Closed subsets of } \mathbb{A}^n\}$$

is a bijection with inverse $V \mapsto I(V)$.

From now on we shall always assume that k is algebraically closed. Why is this called Nullstellensatz?

Corollary 3.8. *Assume that $\mathfrak{a} \subsetneq k[X_1, ..., X_n]$ is a proper ideal. Then there is an element $(x_1, ..., x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$.*

Proof. Since $\mathfrak{a}$ is proper we have $1 \notin \mathfrak{a}$. But then also $1 \notin \sqrt{\mathfrak{a}}$. Thus also $\mathfrak{a}$ is proper. Therefore we get from the injectivity of V that

$$V(\sqrt{\mathfrak{a}}) \neq V(k[X_1, ..., X_n]) = \emptyset \ . \qquad \square$$

Corollary 3.9. *Assume that we are given a finite set of polynomials $f_1, ..., f_k \in k[X_1, ..., X_n]$. Then exactly one of the following things happen:*

(1) *The polynomials have a common zero in k^n.*
(2) *There are polynomials $g_1, ..., g_k \in k[X_1, ..., X_n]$ such that*

$$g_1 f_1 + ... + g_k f_k = 1 \ .$$

Proof. First note that these two alternatives clearly exclude each other. By the last corollary we see that either $(f_1, ..., f_k)$ is the full ideal (which is the second alternative) or that the polynomials have a common zero. $\square$

3.1. Proof of the Nullstellensatz. The next statement is true for any field, even if it is not algebraically closed:

Lemma 3.10 (sometimes also called Nullstellensatz or Zariski-Lemma). *Assume that a field extension $k \to K$ is obtained as $K = k[\alpha_1, ..., \alpha_n]$ by ring-adjoining finitely many elements, i.e. finitely generated as a k-algebra. Then it is algebraic.*

Proof. For $n = 1$ it is clear, as it exactly says that $K = k[\alpha_1]$. Recall that the argument goes as follows: we need to have a relation

$$\alpha_1^{-1} = p(\alpha_1)$$

Thus $p(\alpha_1)\alpha_1 - 1 = 0$ so that α_1 is algebraic.

For general n we consider

$$k_1 := k(\alpha_1) \subseteq K$$

Then K is generated by $(n-1)$-elements over k_1 and we get by induction on n that it is algebraic over k_1. If α_1 is algebraic over k then we are done, since then k_1 is algebraic and thus the rest follow by transitivity of algebraic extensions. Thus we can assume that α_1 is transcendental, i.e. $k_1 = k(\alpha_1)$ is rational functions in α_1. To stress this fact we will write $\alpha_1 = x$ and have $k_1 = k(x)$. We now want to derive a contradiction to finish the proof.

Since K is algebraic and finitely generated over $k(x)$ we can choose a basis $(v_0, ..., v_t)$ of K over $k(x)$. We write down the multiplication table

$$v_i v_j = \sum \frac{p_{ijk}(x)}{q_{ijk}(x)} v_k \tag{1}$$

with $p_{ijk}(x), q_{ijk}(x) \in k[x]$ polynomials. For convenience we set $\alpha_0 = 1$ and write

$$\alpha_s = \sum \frac{f_{st}(x)}{g_{st}(x)} v_t \tag{2}$$

We now want to show that the k-algebra $A = k[\alpha_1, ..., \alpha_n] = k[\alpha_0, \alpha_1, ..., \alpha_n]$ is smaller than K and thus derive a contradiction to the assumption of the lemma that $A = K$.

Every element $a \in A$ is a k-linear combination of products of the α_i's. If we expand using (2) we see that a is a $k(x)$-linear combination of products of the v_i where all the denominators of the coefficients are products of the g_{st}'s. Then using (1) we see that a is a $k(x)$-linear combination of the v_i's where the denominators of the coefficients are products of the $g's$ and $q's$. In particular we see that if we express a as a $k(x)$-linear combination of our basis, then all the coefficients have the property that they can be expressed with a denominator whose irreduzible factors are among the irreducible factors of the polynomials q_{ijk} and g_{st}.
But then the we simply choose another irreducible polynomial $w \in k[x]$ (there are infinitely many) and thus $1/w \in K$ cannot lie in A.[2] $\square$

COROLLARY 3.11. *Assume that k is algebraically closed and R is a non-trivial k-algebra (i.e. a ring with a map $k \to R$) which is finitely generated (i.e. obtained by ring adjoining finitely many elements). Then there is a map $R \to k$ over k.*

PROOF. We choose a maximal ideal $\mathfrak{m} \subsetneq R$ and consider the map $R \to R/\mathfrak{m} = K$. Then by the previous lemma K is algebraic over k. But since k is algebraically closed the map $k \to K$ is an isomorphism, and we find a map as desired. $\square$

Before we prove the next result let us recall that one can invert elements in rings: given a ring R and an element $r \in R$ we can form a new ring $R[r^{-1}]$ with a map $R \to R[r^{-1}]$ with the following universal property: for any other ring S the restriction along $R \to R \to R[r^{-1}]$ induces a bijection

$$\left\{ \text{maps } R[r^{-1}] \to S \right\} \xrightarrow{\simeq} \left\{ \text{maps } R \xrightarrow{f} S \text{ s.t. } f(r) \text{ is a unit} \right\} .$$

Concretely one can construct $R[r^{-1}]$ by considering 'fractions' in R with denominator powers of r or a little bit slicker as

$$R[r^{-1}] = R[T]/(rt - 1) .$$

This presentation also shows that $R[r^{-1}]$ is a finitely generated R algebra.

WARNING 3.12. *Note that in general the map $R \to R[r^{-1}]$ is not injective. The kernel consists exactly of those elements $s \in R$ for which there exists an n such that $r^n s = 0$. This can be deduced directly from the concrete description of $R[r^{-1}]$ as fractions (obviously those elements have to map to zero). In particular we conclude that $R[r^{-1}]$ is the zero ring precisely if r is nilpotent.*

PROOF OF HILBERT'S NULLSTELLENSATZ (THEOREM 3.7). We will prove that for every radical ideal $\mathfrak{a}$ we have that

$$IV(\mathfrak{a}) = \mathfrak{a} . \tag{3}$$

This shows that $IV = \text{id}$. But we have already seen that V is surjective, so that this finishes the proof.
In order to prove (3) we first note that for $f \in \mathfrak{a}$ we have by assumption that for each $x \in V(\mathfrak{a})$ that $f(x) = 0$. Thus $f \in IV(\mathfrak{a})$ and we get the inclusion $\supseteq$. Conversely assume that $f \in IV(\mathfrak{a})$. Consider the ring

$$R = (k[X_1, ..., X_n]/\mathfrak{a}) \left[f^{-1}\right] .$$

[2] Here we use that there are always infinitely many irreducible polynomials over any field, a proof of this fact will be given in the exercises.

which is a finitely generated k-algebra. If $R = 0$ then there would be an $n > 0$ such that $f^n \in \mathfrak{a}$. But then $f \in \mathfrak{a}$ since $\mathfrak{a}$ is radical, and we are done. If $R \neq 0$ then we find by the previous corollary a homomorphism $R \to k$ and thus elements $(x_1, ..., x_n) \in V(\mathfrak{a})$ such that $f(x_1, ..., x_n) \neq 0$. But this contradicts the assumption that $f \in IV(\mathfrak{a})$. □

Corollary 3.13. *Every maximal ideal of* $k[X_1, ..., X_n]$ *is of the form*

$$I(\{\alpha_1, ..., \alpha_n\}) = (X_1 - \alpha_1, ..., X_n - \alpha_n)$$

for an element $(\alpha_1, ..., \alpha_n) \in k^n$.

Proof. For every maximal ideal $\mathfrak{m}$ there is a common zero $(\alpha_1, ..., \alpha_n)$ according to the Nullstellensatz. This means that the ideal is contained in the ideal of all polynomials that vanish at $\alpha_1, ..., \alpha_n$ and hence the two are equal (the latter is certainly not everything). But the latter is exactly the ideal $(X_1 - \alpha_1, ..., X_n - \alpha_n)$, which is clearly maximal as the quotient is k. □

4. The category of affine Varieties

We now want to say things in more categorical language. As before we let k be an algebraically closed field.

Definition 4.1. *Let* $V \subseteq \mathbb{A}^n$ *be a closed subset (i.e. an affine algebraic set). Then we define the* ring of functions *on* V *to be the ring*

$$\mathcal{O}(V) := k[X_1, ..., X_n]/I(V) \ .$$

Example 4.2. We have that $\mathcal{O}(\mathbb{A}^n) = k[X_1, ..., X_n]$. We think of $\mathcal{O}(\mathbb{A}^n)$ as functions on $\mathbb{A}^n$. In general, every element $[p] \in \mathcal{O}(V)$ should be interpreted as a function $V \to k$ represented by the polynomial p. The point is that two polynomials $p, q \in k[X_1, ..., X_n]$ agree on V precisely if they differ by an element in $I(V)$, i.e. represent the same class in $\mathcal{O}(V)$.

Definition 4.3. *A ring* R *is reduced if it does not admit any nilpotent elements, i.e. if* $x^n = 0$ *implies that* $x = 0$.

Proposition 4.4. *The ring* $\mathcal{O}(V)$ *is a finitely generated and reduced* k*-algebra. Conversely any finitely generated, reduced* k*-algebra is of the form* $\mathcal{O}(V)$ *for some affine algebraic set* $V \subseteq k^n$.

Proof. Any finitely generated algebra R is of the form $k[X_1, ..., X_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ (where n is the number of generators). It is reduced precisely if $\mathfrak{a}$ is radical. □

Now we define morphisms of affine algebraic sets. These are informally given by maps that can be given by polynomials (think of our example of smooth manifolds, where morphisms would be smooth maps).

Definition 4.6. *Let* $V \subseteq \mathbb{A}^n$ *and* $W \subseteq \mathbb{A}^m$ *be affine algebraic sets. Then a morphism* $V \to W$ *is given by a map of sets* $f : V \to W$ *for which there are polynomials* $p_1, ..., p_m \in k[X_1, ..., X_n]$ *such that*

$$f(x) = (p_1(x), ..., p_m(x)) \ .$$

for all $x \in V$. *This notion of morphisms is clearly closed under composition and contains all identities. Thus affine algebraic sets and morphisms for a category which we denote by* AffVar_k.

WARNING 4.7. *The polynomials $p_1, ..., p_m$ are not uniquely determined by the map f. They are not part of the datum of a morphism, only their existence is required.*

EXAMPLE 4.8. For $V \subseteq \mathbb{A}^n$ the set $\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^1)$ consist of functions $V \to k$ that can be written as polynomials $p \in k[X_1, ..., X_n]$. Thus by the comparison with Example 4.2 we get a canonical bijection

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^1) = \mathcal{O}(V) \ .$$

so that $\mathcal{O}(V)$ indeed consists of functions.

EXAMPLE 4.9. More generally we see that a map from V to $\mathbb{A}^m$ can be represented by polynomials $p_1, ..., p_m \in k[X_1, ..., X_m]$ modulo $I(V)$, i.e.

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^m) = \mathcal{O}(V)^m \ .$$

From Example 4.8 we in particular see that for a given morphism $f : V \to W$ we get an induced map

$$\mathcal{O}(W) \to \mathcal{O}(V)$$

by pullback (or precomposition) which is easily verified to be a map of rings as follows: we find $(p_1, ..., p_m)$ representing f. Then an element $g \in \mathcal{O}(W)$ is send to the polynomial $[g(p_1, ..., p_m)] \in \mathcal{O}(V)$. This assignment obviously preserves products and sums (of g's). Together this defines a contravariant functor

$$\mathcal{O} : \mathrm{AffVar}_k^{\mathrm{op}} \to \mathrm{Alg}_k^{\mathrm{f.g.,red.}}$$

to the category of finitely generated, reduced k-algebras.

THEOREM 4.10. *This functor is an equivalence of categories.*

PROOF. We have already seen that it is essentially surjective by Proposition 4.4. Thus we only have to show that it is fully faithful, i.e. that the induced map

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V, W) \to \mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W), \mathcal{O}(V))$$

is a bijection. Explicitly this maps the class $[p_1, ..., p_m]$ to the morphism obtained by precomposition as described above. In the special case $W = \mathbb{A}^m$, this map is a bijection as explained in Example 4.9: An element on the left hand side is exactly given by an m-tuple of elements $([p_1], \ldots, [p_n]) \in \mathcal{O}(V)^m$, and the map takes it to the homomorphism $k[X_1, \ldots, X_m] \to \mathcal{O}(V)$ which takes $X_i \mapsto [p_i]$.
For arbitrary $W \subseteq \mathbb{A}^m$ we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{AffVar}_k}(V, W) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^m) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W), \mathcal{O}(V)) \arrow[r, hook] & \mathrm{Hom}_{\mathrm{Alg}_k}(k[x_1, \ldots, x_m], \mathcal{O}(V))
\end{tikzcd}$$

where the upper horizontal arrow exhibits $\mathrm{Hom}_{\mathrm{AffVar}_k}(V, W)$ as subset of those maps represented by polynomials $(p_1, \ldots, p_m)$ which map V to W, and the lower horizontal map exhibits $\mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W), \mathcal{O}(V))$ as subset of those maps which vanish on $I(W)$. Since the upper right composite is injective, the left vertical map is also injective. To see that it is an isomorphism, it therefore suffices to check that it is surjective. Thus we have to see that $[p_1, \ldots, p_m] \in \mathrm{Hom}_{\mathrm{AffVar}}(V, \mathbb{A}^m)$ take V to W if the corresponding map $k[X_1, \ldots, X_m] \to \mathcal{O}(V)$, $X_i \mapsto p_i$, vanishes on $I(W)$. The latter condition spelled out means that for each $f \in I(W)$ we have that $f(p_1, ..., p_n)$ has to lie in $I(V)$. In particular if $x \in V$ then $f(p_1(x), ..., p_n(x)) = 0$, i.e. $p(x) \in W$. $\square$

COROLLARY 4.11. *Two affine algebraic sets V and W are isomorphic precisely if their rings of functions $\mathcal{O}(V)$ and $\mathcal{O}(W)$ are isomorphic.*

5. Irreducible Varieties

COROLLARY 5.1 (of the Nullstellensatz). *Let V be an affine algebraic set over an algebraically closed field. Then the assigment*

$$v \in V \mapsto \{[f] \in \mathcal{O}(V) \mid f(v) = 0\}$$

is a bijection between points of V and maximal ideals of $\mathcal{O}(V)$.

PROOF. Ideals in $\mathcal{O}(V)$ are given by ideals in $k[X_1, ..., X_n]$ that contain $I(V)$. Thus the maximal ones are precisely the maximal ideals of $k[X_1, ..., X_n]$ which contain $I(V)$, i.e. points $v \in \mathbb{A}^n$ such that $v \in V$. □

Now since $\mathbb{A}^n$ is a topological space, we get a subspace topology on $V \subseteq \mathbb{A}^n$ in which a subset

$$V' \subseteq V$$

is closed precisely if $V' \subseteq \mathbb{A}^n$ is closed, i.e. V' is itself an affine algebraic set contained in V. Thus we get:

PROPOSITION 5.2. *The closed subsets of V are in one-to-one correspondence to radically closed ideals in $\mathcal{O}(V)$.*

PROOF. We simply observe that ideals in $\mathcal{O}(V)$ correspond to ideals in $k[X_1, ..., X_n]$ which contain $I(V)$. Under this correspondence, implemented by taking preimages, we see that radically closed ideals correspond to radically closed ideals since

$$(k[X_1, ..., X_n]/I(V))\,/\mathfrak{a} = k[X_1, ..., X_n]/\mathfrak{a}'$$

□

DEFINITION 5.3. *A topological space X is called* irreducible *if it can not be written as the union of two closed subspaces (neither of which agrees with X).*
An affine algebraic set $V \subseteq \mathbb{A}^n$ is called irreducible (or affine variety, but we try to avoid this notions here to not create confusion), if it is irreducible when considered as a topological space.

Equivalently, V is irreducible if we cannot write it as $V = V_0 \cup V_1$ for affine algebraic subsets $V_0, V_1 \subsetneq V$. Also equivalent: if $V \subseteq V_0 \cup V_1$ for $V_0, V_1 \subseteq \mathbb{A}^n$ then either $V \subseteq V_0$ or $V \subseteq V_1$.

EXAMPLE 5.4. The affine algebraic set

$$\{(x, y) \in \mathbb{C}^2 \mid x^2 - y^2 = 0\}$$

is not irreducible, since we can write it as the union of

$$V_0 = \{(x, y) \in \mathbb{C}^2 \mid x - y = 0\} \qquad \text{and} \qquad V_1 = \{(x, y) \in \mathbb{C}^2 \mid x + y = 0\}\ .$$

EXAMPLE 5.5. The affine algebraic set $k = \mathbb{A}^1$ is irreducible, since each proper closed subset is finite (solutions to a polynomial equation since every ideal is principal) and k is infinite, being algebraically closed. Thus it cannot be written as union of two closed subsets.

PROPOSITION 5.6. *For an affine algebraic set $V \subseteq \mathbb{A}^n$ the following are equivalent:*

(1) *V is irreducible*
(2) *$I(V) \subseteq k[X_1, ..., X_n]$ is a prime ideal*
(3) *$\mathcal{O}(V)$ is a domain, i.e. does not have zero divisors.*

PROOF. (2) and (3) are clearly equivalent. Thus we have to show that (1) is equivalent to (2). To this end note that V is reducible precisely if we have $V = V_0 \cup V_1$ for proper subvarieties V_0, V_1. This equivalently means that we have

$$I(V) = I(V_0 \cup V_1) = I(V_0) \cdot I(V_1)$$

for the vanishing ideals $I(V), I(V_0), I(V_1) \subseteq k[X_1, ..., X_n]$ with $I(V) \subsetneq I(V_i)$. Now the claim follows from the assertion that a radically closed ideal $\mathfrak{a} \subseteq k[X_1, ..., X_n]$ is prime exactly if it can not be written as a product of two strictly larger radical ideals $\mathfrak{b}$ and $\mathfrak{c}$.
If

$$\mathfrak{a} = \mathfrak{b} \cdot \mathfrak{c}$$

then we find $b \in \mathfrak{b}$ and $c \in \mathfrak{c}$ such that $bc \in \mathfrak{a}$ but neither b nor c is in $\mathfrak{a}$. Thus $\mathfrak{a}$ is not prime. If $\mathfrak{a}$ is not prime then we find $b, c \in k[X_1, ..., X_n]$ with $bc \in \mathfrak{a}$ but neither of the two elements is in $\mathfrak{a}$. But then we consider

$$\mathfrak{b} := \sqrt{\mathfrak{a} + (b)} \qquad \text{and} \qquad \mathfrak{c} := \sqrt{\mathfrak{a} + (c)}$$

We have that $\mathfrak{b} \cdot \mathfrak{c} = \mathfrak{a}$ as one easily verifies, but both are larger than $\mathfrak{a}$. □

COROLLARY 5.7. *$\mathbb{A}^n(k)$ is irreducible.*

PROOF. The polynomial ring is a domain. □

EXAMPLE 5.8. For every prime polynomial $p \in k[X_1, ..., X_n]$ the hypercurve $\{p = 0\}$ is irreducible.

DEFINITION 5.9. *A topological space X is called noetherian, if every descending chain*

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq ...$$

of closed subsets becomes eventually constant.

LEMMA 5.10. *Every affine algebraic set V is noetherian as a topological space, i.e. every descending chain of affine algebraic sets*

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq ...$$

becomes constant.

PROOF. Clear since $\mathcal{O}(V)$ is noetherian as a quotient of $k[X_1, ..., X_n]$. □

DEFINITION 5.11. *Let X be a topological space. A subset $V \subseteq X$ is called irreducible if it is irreducible in the subspace topology, i.e. for every inclusion $V \subseteq V_0 \cup V_1$ with V_i closed we have $V \subseteq V_0$ or $V \subseteq V_1$. An irreducible component is a maximal irreducible subset of X.*

LEMMA 5.12. *Irreducible components are closed and every irreducible subset is contained in an irreducible component.*

PROOF. For the first part we note that the closure of an irreducible set is irreducible. This is clear by definition. Thus if V is maximal it has to agree with its closure. For the second part assume that we are given an irreducible subset $M \subseteq X$. Then the set of all irreducible subsets between M and X is inductively ordered (i.e. every chain has an upper bound): given a sequence

$$M \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$$

we can consider the unions $M_\infty = \bigcup M_i$. This is irreducible, since for a given decomposition $M_\infty = A_0 \cup A_1$ we get that since all the M_i's are irreducible each of them has to be contained in A_0 or A_1. Thus we find infinitely many that lie in one of them, say A_0. But then also the union does. Thus by Zorn's Lemma a maximal element exists, i.e. M is contained in an irreducible component.

□

PROPOSITION 5.13. *Let X be a noetherian topological space. Then it admits finitely many, uniquely determined irreducible components $V_1, ..., V_r$. We have*

$$X = V_1 \cup .... \cup V_r \ .$$

The space X is not the union of $r-1$ of them.
Conversely if we can write X as such a union of closed irreducible subsets such that X is not the union of $r-1$ of them (equivalently: they do not contain each other) then these are already the irreducible components.

PROOF. If X is not irreducible, then we can write it inductively as the unions of finitely many irreducible closed subspaces (using that the space is noetherian)

$$X = V_0 \cup ... \cup V_r.$$

We can WLOG assume that neither of those is contained in another one. Now any irreducible subset $M \subseteq X$ has to be contained in one of the V_i by definition of irreducible. If M is an irreducible component, then it has to agree with the respective V_i. Thus every irreducible component has to agree with one of the V_i. Since every V_i is contained in an irreducible component this also shows that all of the V_i are irreducible components. □

Note that irreducible components of V correspond to prime ideals in $\mathcal{O}(V)$. Thus the last result can be translated into the existence of minimal prime ideals in this ring.

EXAMPLE 5.14. The irreducible components of

$$\{(x, y) \in k^2 \mid xy = 0\}$$

are given by the axes $\{x = 0\}$ and $\{y = 0\}$ since V is the union of those but not of less of them.

EXAMPLE 5.15. More generally for any polynomial $p \in k[X_1, ..., X_n]$ we consider the hypercurve

$$V = \{x \in \mathbb{A}^n \mid p(x) = 0\} = \{p = 0\} \ .$$

If we factor p into prime factors $p = p_1^{n_1} \cdot ... \cdot p_k^{n_k}$ then we first observe that with $p' = \prod p_i$ we have $V = \{p' = 0\}$ In fact, $(p') = \sqrt{(p)}$. But now we also have

$$V = \{p_1 = 0\} \cup \{p_2 = 0\} \cup \ldots \cup \{p_k = 0\}$$

Again these are clearly not contained in each other (as the prime polynomials cannot divide each other).

We finish this section by the following table

Geometry	Algebra
V: affine algebraic set	$\mathcal{O}(V)$: reduced, finitely generated k-algebra
$x \in V$ point	$I(x)$ maximal ideal in $\mathcal{O}(V)$
$V' \subseteq V$ closed subsets	$I(V')$ radical ideals in $\mathcal{O}(V)$
irreducible subvarieties	prime ideals
irreducible components	minimal prime ideals

6. The spectrum of a ring

Recall that the points of an affine algebraic set V are the maximal ideals in $\mathcal{O}(V)$. We now want to do this for general rings A replacing $\mathcal{O}(V)$.

Definition 6.1. *Let A be a commutative ring. Then we define the* Spectrum *of A, denoted as* $\mathrm{Spec}(A)$*, as the set of prime ideals of A (remember that prime ideals are not the full ring). The maximal spectrum is the subset*

$$\mathrm{mSpec}(A) \subseteq \mathrm{Spec}(A)$$

consisting of the maximal ideals. For a given point $x \in \mathrm{Spec}(A)$ we define the residue field at x *to be*

$$\kappa(x) := \mathrm{Frac}(A/x)$$

the fraction field of the domain A/x. [3] *An element $f \in A$ gives rise to an element*

$$f(x) \in \kappa(x)$$

represented by the class $[f]$.

We informally want to think of an element $f \in A$ as a function on $\mathrm{Spec}(A)$ which takes x to an element in the residue field $\kappa(x)$, thus the domain varies. We clearly have that for $f, g \in A$ that $(f \cdot g)(x) = f(x)g(x)$ and $(f + g)(x) = f(x) + g(x)$.

Example 6.2. If $A = \mathcal{O}(V)$ for an affine algebraic set V over an algebraically closed field k, then $V \cong \mathrm{mSpec}(A)$ and for every $x \in V$ we have that $\kappa(x) \cong k$, the isomorphism is given by

$$\mathcal{O}(V)/I(x) \to k \qquad [f] \mapsto f(x).$$

Thus for a given $f \in \mathcal{O}(V)$ we get that under this identification $f(x)$ is simply the evaluation of f at the point x!
The spectrum $\mathrm{Spec}(A)$ now consists of the prime ideals, which are the irreducible closed subsets of V. For a given point $x \in \mathrm{Spec}(A)$ corresponding to an irreducible variety $V' \subseteq V$ the fraction field $\kappa(x)$ is given by the fraction field of the (domain) $\mathcal{O}(V')$. The evaluation $f(x)$ for an element $f \in \mathcal{O}(V)$ at x is then given by considering the restriction of f to V' as an element in the fraction field of $\mathcal{O}(V')$.

Example 6.3. Let's consider the case $V = \mathbb{A}^1$, i.e. $A = \mathcal{O}(V)$. Then $\mathrm{Spec}(A)$ consists of all the points of V corresponding to the maximal ideals and additionally the prime ideals. But the ring $k[x]$ has only one prime ideal that is not maximal, namely 0. Geometrically this corresponds to the irreducible variety V itself. Thus $\mathrm{Spec}(A)$ has the points $\mathbb{A}^1$ and one additional point 0, which is morally 'arbitrarily close' to any other point. We will make the latter precise soon.

[3] That means we invert every nonzero element in A/x. Note that for a maximal ideal x we have that A/x is already a field, thus $\kappa(x) = A/x$

Example 6.4. Let's consider the case $V = \mathbb{A}^2$, i.e. $A = \mathcal{O}(V)$. Then $\mathrm{Spec}(A)$ consists of all the points of V corresponding to the maximal ideals and additionally the points corresponding to irreducible subsets $V' \subseteq V$ that are not points. There is again a point for the full subset V (corresponding to the 0-ideal) and for every (normalized) irreducible polynomial $p \in k[X_1, X_2]$ there is a further point corresponding to the zero locus $\{p = 0\}$. One can show that these are all points (but we will not do this here).

Example 6.5. Consider the ring $A = \mathbb{Z}$. Then the prime ideals are given by $p\mathbb{Z}$ for p a prime and 0. The maximal ideals among them are the $p\mathbb{Z}$, i.e. everything except 0 is maximal. Thus $\mathrm{Spec}(\mathbb{Z})$ has a point for every prime and an additional point 0, which one should think of a point which is 'arbitrarly close to all other points'. The residue fields are given by

$$\kappa(p) = \mathbb{F}_p$$

and

$$\kappa(0) = \mathbb{Q}$$

An element $n \in \mathbb{Z}$ acts as the function of $\mathrm{Spec}\,\mathbb{Z}$ which sends n to the value n considered in each of these fields.

Example 6.6. For any field k we have that $\mathrm{Spec}(k)$ only consist of a single point (corresponding to the zero ideal) and the fraction field is k itself.

Proposition 6.7. *For any map $A \to B$ of rings the induced map*

$$\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \qquad \mathfrak{p} \mapsto f^{-1}\mathfrak{p}$$

is well-defined.

Proof. We only have to show that the ideal $f^{-1}\mathfrak{p}$ is prime. Thus assume $ab \in f^{-1}\mathfrak{p}$. Then $f(a)f(b) = f(ab) \in \mathfrak{p}$, therefore either $f(a)$ or $f(b)$ in $\mathfrak{p}$, which implies that either a or b are in $f^{-1}\mathfrak{p}$. $\square$

Warning 6.8. *While* $\mathrm{mSpec}(A)$ *seems in some sense more natural than* $\mathrm{Spec}(A)$ *it is not functorial. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ induces the map*

$$\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$$

which sends (0) *to* (0)*. But* $\mathrm{mSpec}(\mathbb{Q}) = \mathrm{Spec}(\mathbb{Q})$ *while* $\mathrm{mSpec}(\mathbb{Z})$ *does not contain the zero point. Said differently: the pullback of a maximal ideal is in general only a prime ideal but not maximal.*

Definition 6.9. *Let M be a subset of a ring A. Then we define the vanishing locus of M as*

$$V(M) = \{x \in \mathrm{Spec}(A) \mid f(x) = 0\ \forall f \in M\} = \{x \in \mathrm{Spec}(A) \mid M \subseteq x\}\ .$$

We leave the equality as an exercise to the reader to get used to the definitions.

Proposition 6.10.
(1) *We have that $V(M) = V(\sqrt{(M)})$.*
(2) *There is a topology on* $\mathrm{Spec}(A)$ *called the* Zariski topology *such that the closed sets are exactly the vanishing loci $V(M)$.*
(3) *For any ring map $f : A \to B$ the map* $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is continuous.*[4]

[4]That is preimages of open sets are open, or equivalently preimages of closed sets are closed.

(4) *For any ideal* $\mathfrak{a}$ *the map*

$$\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$$

induced by the quotient map is a homoeomorphism onto the closed subset $V(\mathfrak{a})$.

Proof. (1) is clear. The proof of (2) proceeds exactly as in Proposition 3.1:

$$\bigcap_{i\in I} V(M_i) = V(\cup_{i\in I} M_i) \qquad V(M) \cup V(N) = V(M \cdot N).$$

For (3) note that

$$\mathrm{Spec}(f)^{-1}(V(M)) = \{x \in \mathrm{Spec}(B) \mid M \subseteq f^{-1}(x)\} = V(f(M)) \ .$$

For (4) we note that the map $A \to A/\mathfrak{a}$ certainly induces an bijection $\mathrm{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a})$. For any ideal over $\mathfrak{a}$ we have that under this bijection $V(\bar{\mathfrak{b}})$ corresponds to $V(\mathfrak{b})$. ☐

Theorem 6.11. *Closed subsets of* $\mathrm{Spec}(A)$ *are in 1-1 correspondence with radically closed ideals of* A. *The correspondence sends an Ideal* $\mathfrak{a}$ *to the vanishing locus and conversely sends a closed subset* $V \subseteq \mathrm{Spec}(A)$ *to the ideal*

$$\{f \in A \mid f(v) = 0 \ \forall v \in V\} = \bigcap_{\mathfrak{p}\in V} \mathfrak{p}$$

Proof. By definition and the previous proposition, each closed subset is of the form $V(\mathfrak{a})$ for a radically closed ideal $\mathfrak{a}$. Thus it suffices to show that for each radically closed ideal $\mathfrak{a}$ we have that

$$\mathfrak{a} = \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \mathfrak{p} \ .$$

We will more generally show that for each ideal $\mathfrak{a}$ (not necessarily radically closed) in a commutative ring A we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \mathfrak{p} \ .$$

By replacing A by $A/\mathfrak{a}$ this is equivalent to the statement that for every ring A we have

$$\sqrt{0} = \bigcap_{\mathfrak{p}} \mathfrak{p}$$

i.e. that for any ring (commutative) the ideal of nilpotent elements is the intersection of all prime ideals. We certainly have that every nilpotent element is contained in each prime ideal. Thus it suffices to find for a non-nilpotent element $x \in A$ a prime ideal which does not contain x. To this end we consider the ring $A[x^{-1}]$ which is not zero, since x is not nilpotent. Then we choose a maximal ideal $\mathfrak{m}$ in this ring and consider the prime ideal $i^{-1}\mathfrak{m}$ where $i : A \to A[x^{-1}]$ is the canonical map. This does not contain x. ☐

Remark 6.12. This implies similarly to the Nullstellensatz (Corollary 3.8) that for every ideal $\mathfrak{a} \subsetneq A$ we have a point $x \in \mathrm{Spec}(A)$ with $f(x) = 0$ for all $f \in \mathfrak{a}$. But one can do better: for every $\mathfrak{a} \neq A$ we find a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. Thus $\mathfrak{m} \in V(\mathfrak{a})$, i.e. all $f \in \mathfrak{a}$ vanish on $\mathfrak{m}$, i.e. we even have a zero in the maximal spectrum.

Recall that a space X is called compact or quasi-compact (if one wants to emphasize the fact that it is not Hausdorff) if for every open cover $X = \bigcup_{i \in I} U_i$ there is a finite set of open sets $U_{i_0}, ..., U_{i_n}$ such that $X = U_{i_0} \cup ... \cup U_{i_n}$. In words: every cover has a finite subcover. Equivalently: if there are closed sets V_i with $\cap V_i = \emptyset$ then there are already finitely many whose intersection is empty.

PROPOSITION 6.13. *The topological space* $\mathrm{Spec}(A)$ *is quasi-compact.*

PROOF. Assume that

$$\emptyset = \bigcap_{i \in I} V(M_i) = V(\cup_{i \in I} M_i)$$

If $\mathfrak{a}$ is the ideal generated by all the M_i then $V(\mathfrak{a}) = \emptyset$ so that $\mathfrak{a} = A$. But this means that $1 \in A$. But then 1 is already contained in the ideal generated by finitely many M_i and thus the intersection of those $V(M_i)$ is already empty. $\square$

For a subset $M \subseteq A$ we denote the open set

$$D(M) = V(M)^c \subseteq \mathrm{Spec}(A) \ .$$

This is the set of all $x \in \mathrm{Spec}(A)$ with $f(x) \neq 0$ for at least one $f \in M$.

PROPOSITION 6.14. *Given $f \in A$ the map*

$$\mathrm{Spec}(A[1/f]) \to \mathrm{Spec}(A)$$

induced by the canonical map $i : A \to A[1/f]$ is an open embedding with image $D(f)$.

PROOF. Prime ideals in $A[1/f]$ are in one-to-one correspondence with isomorphism classes of quotients $A[1/f] \to C$ which are domains. These are in one-to-one correspondence with domain quotients $A \to B$ which send f to a nonzero element (by localizing / passing to the image of A). So we see that prime ideals in $A[1/f]$ are in one-to-one correspondence with prime ideals of A which do not contain f (and the bijection is given by intersecting an ideal of $A[1/f]$ with A, i.e. by $\mathrm{Spec}(i)$). This tells us that $\mathrm{Spec}(i)$ is a bijection from $\mathrm{Spec}(A[1/f])$ onto the complement of $V(f)$ in $\mathrm{Spec}(A)$. Thus it remains to prove that $\mathrm{Spec}(i)$ is an embedding, i.e. that $\mathrm{Spec}(A[1/f])$ carries the subspace topology. Since $\mathrm{Spec}(i)$ is continuous, the intersection of a closed set in $\mathrm{Spec}(A)$ with $\mathrm{Spec}(A[1/f])$ is closed in $\mathrm{Spec}(A[1/f])$, so it suffices to show that conversely, every closed set in $\mathrm{Spec}(A[1/f])$ arises as intersection with a closed set in $\mathrm{Spec}(A)$.
To this end consider a closed set

$$V(M) \subseteq \mathrm{Spec}(A[1/f])$$

defined by some set $M = \{\frac{p_i}{q_i} \mid i \in I\} \subseteq A[1/f]$. Then we consider the set $M' := \{p_i \mid i \in I\} \subseteq A$. We have that

$$\begin{aligned} V(M') \cap \mathrm{Spec}(A[1/f]) &= \{x \in \mathrm{Spec}(A[1/f]) \mid i(M') \subseteq x\} \\ &= \{x \in \mathrm{Spec}(A[1/f]) \mid M \subseteq x\} = V(M) \ . \end{aligned}$$

This finishes the proof. $\square$

DEFINITION 6.15. *We call an open set $U \subseteq \mathrm{Spec}(A)$ principal open, if it is of the form $D(f) \cong \mathrm{Spec}(A[1/f])$ for some $f \in A$. In the principal case we set*

$$\mathcal{O}(U) = A[1/f]$$

Remark 6.16. Note that $\mathcal{O}(U)$ is well-defined (i.e. only depends on the open set U) since $V(f) = V(f')$ precisely if $\sqrt{(f)} = \sqrt{(f')}$ and then we have such that $f^n = gf'$ and $(f')^m = g'f$ so that inverting f inverts f' and vice versa. Thus we have a unique isomorphism $A[1/f] \cong A[1/f']$.

Proposition 6.17. *Every open set $U \subseteq \mathrm{Spec}(A)$ is a union of principal opens (that is, the principal opens form a basis of the topology) and every finite intersection of principal opens is principal open.*

Proof. We have

$$U = V(M)^c = (\bigcap_{f \in M} V(f))^c = \bigcup_{f \in M} D_f.$$

and

$$D_f \cap D_g = (V(f) \cup V(g))^c = V(fg)^c = D_{fg} \ . \qquad \square$$

7. Generic points

Definition 7.1. *Let X be a topological space and $V \subseteq X$ be a closed subset. A point $x \in V$ is called generic point of V if $\overline{x} = V$. If $V = X$ then it is simply called generic point (of X).*
A space X is called T_0-space if for each pair of points $x \neq y$ there is an open set which contains exactly one of the two points.

If V admits a generic point, then is is irreducible, since points are certainly irreducible and closures preserve irreducibility. If X is T_0 then generic points are, if they exist, unique since if we can separate x and y by an open set then $\overline{x} \neq \overline{y}$.

Example 7.2. For every domain A the point $(0) \in \mathrm{Spec}(A)$ is generic since every closed set $V(M) = \{x \in \mathrm{Spec}(A) \mid M \subseteq x\}$ containing (0) has to be everything. Morally this means that (0) lies infinitely close to any other point in $\mathrm{Spec}(A)$ (as it is contained in each neighborhood). This is for example true for $\mathrm{Spec}(\mathbb{Z})$ which has a bunch of closed points (all the primes) and the point (0). This is also true for $\mathrm{Spec}(k[X])$ which is in fact homeomorphic to $\mathrm{Spec}(\mathbb{Z})$ as long as k is countable.

Definition 7.3. *A space is called* sober *if it is T_0 and every irreducible closed subset has a generic point.*

Example 7.4. Every Hausdorff space is sober (exercise). The space $\mathbb{A}^n = k^n$ with the Zariski topology is not sober, since it is irreducible but does not posses a generic point (all points are in fact closed).
Every finite T_0-space is sober. To see this it suffices to show that a finite, irreducible T_0 space has a generic point. If not, then it has to be the unions of the closures of its points, but not contained in any of them in contradiction to irreducibility.

Proposition 7.5. *For any ring A the space* $\mathrm{Spec}(A)$ *is sober.*

Proof. Assume that $x \neq y$ are two distinct points in $\mathrm{Spec}(A)$. Then there has to be an element $f \in A$ which is contained in one of the two ideals, but not the other. But then the open set $D_f \subseteq \mathrm{Spec}(A)$ which is the subset of all prime ideals which do not contain the element f containes exactly one of the two prime ideals.
Assume that $V(M)$ is an irreducible closed subset. Then we can replace M by a radical ideal $\mathfrak{a}$ so that we assume $V(\mathfrak{a})$ is irreducible. This however implies that for two elements $f, g \in A$ with $fg \in \mathfrak{a}$ we have $V(\mathfrak{a}) \subseteq V(fg) = V(f) \cup V(g)$. Thus

either f or g are in $\mathfrak{a}$. Thus $\mathfrak{a}$ is prime. We have shown that irreducible closed subsets are of the form $V(\mathfrak{p})$ for $\mathfrak{p}$ prime. But then we have that $\mathfrak{p} \in V(\mathfrak{p})$ and that $V(\mathfrak{p})$ is the closure of the point $\mathfrak{p}$ since every closed subset that contains $\mathfrak{p}$ also contains $V(\mathfrak{p})$. Thus $\mathfrak{p}$ is a generic point. □

Remark 7.6. We have now seen that $\mathrm{Spec}(R)$ is quasi-compact, sober and admits a basis for the topology closed under intersections of open subsets that are also quasi-compact (since they are homeomorphic to $\mathrm{Spec}(R[1/f])$). It is a theorem of Hochster that the topological spaces of the form $\mathrm{Spec}(R)$ are precisely the spaces which have these properties, i.e. are quasi-compact, posses a basis of quasi-compact opens which are closed under intersections and are sober. Such spaces are called *spectral spaces.*

Geometry	Algebra
$\mathrm{Spec}(A)$	A any ring
$x \in \mathrm{Spec}(A)$	Prime ideals
closed points	maximal ideals
$V \subseteq \mathrm{Spec}(A)$ closed	radically closed ideals
irreducible closed subsets	prime ideals

Construction 7.7. Assume X is any topological space. Then we can form a new topological space $S(X)$ called the sobrification which comes with a continuous map $X \to S(X)$, is sober and initial among such. Concretely we have that

$$S(X) = \{V \subseteq X \mid V \text{ irreducible and closed}\} .$$

We define a topology on $S(X)$ by letting the closed sets be the subsets of $S(X)$ of the form

$$\{V \in S(X) \mid V \subseteq A\}$$

for closed sets $A \subseteq X$. The map $X \to S(X)$ is given by $x \mapsto \overline{\{x\}}$. We leave it as an exercise to check that this has the desired properties, concretely that:

(1) $S(X)$ is a topological space
(2) The map $X \to S(X)$ is continuous
(3) $S(X)$ is sober
(4) Every continuous map $X \to Y$ with Y sober factors uniquely over $S(X)$
(5) If X is T_0 then the map $X \to S(X)$ is an embedding.

We now have the following immediate consequence:

Theorem 7.8. *Let V be an affine algebraic set over an algebraically closed field k. Then the map*

$$V = \mathrm{mSpec}(\mathcal{O}(V)) \to \mathrm{Spec}(\mathcal{O}(V))$$

is a continuous embedding and exhibits $\mathrm{Spec}(\mathcal{O}(V))$ *as the sobrification of V.*

This in particular shows that in passing from V to the prime spectrum we exactly add 'generic points' of irreducible subvarieties.

Remark 7.9. In general it is not true that $\mathrm{mSpec}(A) \to \mathrm{Spec}(A)$ is a sobrification. This is precisely the case if in the ring A every prime ideal is the intersection of maximal ideals. Such rings are called Jacobson rings. Hilbert's theorem can then be interpreted as saying that quotients of $k[X_1, ..., X_n]$ are Jacobson rings. We will however from now on work with the prime spectrum.

8. The structure sheaf

For a topological space X we can consider the partially ordered set of open subsets of X. We can consider this as a category which we denote $\mathrm{Open}(X)$. Concretely the objects of $\mathrm{Open}(X)$ are the open sets $U \subseteq X$ and the set of morphisms are given by

$$\mathrm{Hom}_{\mathrm{Open}(X)}(U,V) = \begin{cases} \mathrm{pt} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

Every continuous map $f : X \to Y$ gives rise to a functor

$$\mathrm{Open}(Y) \to \mathrm{Open}(X) \qquad U \mapsto f^{-1}U \ .$$

REMARK 8.1. One can try to reconstruct the topological space X from the category $\mathrm{Open}(X)$. It turns out, that this is exactly possible if the space X is sober. The idea is the following: any point $\mathrm{pt} \xrightarrow{x} X$ gives rise to a functor

$$x^{-1} : \mathrm{Open}(X) \to \mathrm{Open}(\mathrm{pt})$$

which preserves arbitrary unions and finite intersections (which can be characterised as colimits and limits in the categories). Conversely one can consider the set of all functors

$$\mathrm{Open}(X) \to \mathrm{Open}(\mathrm{pt})$$

preserving arbitrary unions and finite intersections. It turns out that these are precisely in bijection with closed, irreducible subsets $V \subseteq X$ for any topological space X. Thus the assignment which sends each point x to the associated functor x^{-1} is a bijection precisely if X is sober.

Let $\mathcal{C}$ be any category. Examples to have in mind are $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}, \ldots$

DEFINITION 8.2. *A $\mathcal{C}$-valued* presheaf *on a topological space X is a functor*

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C} \ .$$

We will refer to $F(U)$ as the value on U or as the sections of F over U*. For an inclusion of open sets $U \subseteq V$ we call the induced map $F(V) \to F(U)$ the restriction map (or restriction along $U \subseteq V$). This map is sometimes written as $(-)|_U$ The category of presheaves is the functor category*

$$\mathrm{PSh}(X;\mathcal{C}) := \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C}) \ .$$

EXAMPLE 8.3. We consider the functor

$$F_1(U) := C^0(U, \mathbb{R})$$

with the restriction of functions. This is a presheaf of rings. We also have the presheaf of all set maps

$$F_2(U) := \mathrm{Hom}_{\mathrm{Set}}(U, \mathbb{R})$$

This is also a sheaf of rings and the inclusion of continuous maps in all maps induces a morphism of presheaves $F_1 \to F_2$. We also have the constant presheaf $F_3(U) = \mathbb{R}$ for all U and all restrictions the identity. Thinking of this as constant functions with value in $\mathbb{R}$ we get a further morphism

$$F_3 \to F_1 \ .$$

Definition 8.4. *Assume that the category* $\mathcal{C}$ *has products. A presheaf* $F \in \mathrm{PSh}(X;\mathcal{C})$ *is called a* sheaf *if for each collection* $(U_i)_{i\in I}$ *of open subsets in* X *with* $U = \cup_{i\in I} U_i$ *the induced diagram*

$$F(U) \xrightarrow{(-|_{U_i})_{i\in I}} \prod_{i\in I} F(U_i) \underset{(-|_{U_i\cap U_j}\circ \mathrm{pr}_j)_{i,j\in I}}{\overset{(-|_{U_i\cap U_j}\circ \mathrm{pr}_i)_{i,j\in I}}{\rightrightarrows}} \prod_{i,j\in I} F(U_i\cap U_j)$$

is a limit cone in $\mathcal{C}$*. Here the two maps project to the* i*-th respective* j*-th factor and then restrict to the respective subset.*
The category of sheaves $\mathrm{Shv}(X;\mathcal{C})$ *is the full subcategory of* $\mathrm{PSh}(X;\mathcal{C})$ *spanned by the sheaves.*

Remark 8.5. Assume that $\mathcal{C} = \mathrm{Set}$ (or any category like Ab or Ring consisting of sets with some algebraic extra structure for which limits are computed underlying). Then the sheaf-condition is explicitly saying that for any familiy of elements

$$s_i \in F(U_i)$$

such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ there is a unique element $s \in F(U)$ with $s|_{U_i} = s_i$. Morally this means that one can 'glue' elements from local ones or elements are determined locally. In this case, if the map from $F(U)$ to the equalizer is just injective, we say that F is separated.

Example 8.6. Consider the presheaf C^0. This is a sheaf, since if we have locally defined continuous functions $U_i \to \mathbb{R}$ which agree in double intersections $U_i\cap U_j \to \mathbb{R}$ then we get a unique continuous function $U \to \mathbb{R}$. The important fact is that being continuous can be checked locally (check this as an exercise if you are not sure).
The presheaf of constant functions is in general not a sheaf, since the sheaf condition applied to the *empty* cover of $U =$ requires that

$$F() \longrightarrow \prod_{i\in\emptyset} F(U_i) \rightrightarrows \prod_{i,j\in\emptyset} F(U_i\cap U_j)$$

is a limit cone. Since empty products are simply pt, this precisely means that $F(\emptyset) = \mathrm{pt}$ if F is a sheaf, which is not satisfied by the constant presheaf.
If we fix this by defining instead a presheaf with $F(\emptyset) = \mathrm{pt}$, $F(U) = \mathbb{R}$ for $U \neq \emptyset$, this is still not a sheaf in general: Assume that we have disjoint open subsets $U, V \subseteq X$. Then we can consider constant functions $\lambda_1 : U \to \mathbb{R}$ and $\lambda_2 : V \to \mathbb{R}$. They agree on the intersection $U \cap V = \emptyset$ but there can not be a constant function extending them unless $\lambda_1 = \lambda_2$.

Proposition 8.7. *Assume that* $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ *is a sheaf.*

1. *Then* $F(\emptyset)$ *is terminal*
2. *For disjoint open set* $U, V \subseteq X$ *we have that the map*
$$F(U\cup V) \xrightarrow{(-|_U,-|_V)} F(U)\times F(V)$$
is an isomorphism.
3. *If we have a sequence of open subsets* $U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots$ *with* $\bigcup_{i=0}^{\infty} U_i = U$*, then the map*
$$F(U) \to \varprojlim F(U_i)\ .$$
is an isomorphism.

(4) *More generally assume that we have a map of posets*

$$I \to \mathrm{Open}(X) \qquad i \mapsto U_i$$

where I has infima for pairs of objects and the map sends those to interscs- tions (i.e. preserves infima). Then with $U := \cup U_i$ we have that

$$F(U) = \lim_{I^{\mathrm{op}}} F(U_i)$$

PROOF. The reader that is not yet completely versed in the category theoretic language might want to give these proofs first for the case $\mathcal{C} = \mathrm{Set}$.
1) For the empty set we have the covering consisting of no open subset. Then we get that $F(\emptyset)$ is the equalizer of the two empty products, These are terminal. Thus we get that $F(\emptyset)$ is itself terminal (think about why the equalizer is itself terminal).
2) For $U \cup V$ we take the cover consisting of the two open sets U and V. Applying the sheaf condition we get a limit diagram

$$F(U \cup V) \longrightarrow F(U) \times F(V) \rightrightarrows F(U) \times F(V) \times F(U \cap V) \times F(V \cap U)\ .$$

Here since U and V are disjoint we find that $F(U \cap V) = \mathrm{pt}$ and thus we get that

$$F(U \cup V) \longrightarrow F(U) \times F(V) \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} F(U) \times F(V)\ .$$

But the equaliser of the identity map with itself is simply the object $F(U) \times F(V)$.

3) We consider the covering of U given by the U_i. Then we get that

$$\begin{aligned} F(U) &= \mathrm{Eq}\left(\prod_i F(U_i) \Rightarrow \prod_{i,j} F(U_{\min(i,j)})\right) \\ &= \mathrm{Eq}\left(\prod_i F(U_i) \Rightarrow \prod_{i \le j} F(U_i)\right) \end{aligned}$$

where the equality holds by universal properties (the upper product is strictly larger than the lower one, but the remaining terms are just copies of terms that are already there and the map factors accordingly). (4) works exactly as (3). $\square$

EXAMPLE 8.8. If $\mathcal{B}$ is a basis of the topology of X closed under intersections, we see from (4) that for a sheaf F on X we have

$$F(U) = \lim_{V \subseteq U, V \in \mathcal{B}} F(V)\ .$$

Now using this we shall construct a specific sheaf on $\mathrm{Spec}(A)$ called the structure sheaf and denoted $\mathcal{O}_{\mathrm{Spec}(A)}$ or simply $\mathcal{O}$. It will have the property that

$$\mathcal{O}(D_f) := A[1/f].$$

as in Definition 6.15. In order to turn this into a proper definition we will show that one can define sheaves on a basis of the topology.

DEFINITION 8.9. *Let X be a topological space and $\mathcal{B}$ be a basis of the topology stable under finite intersection. A presheaf on $\mathcal{B}$ is a functor*

$$\mathcal{B}^{\mathrm{op}} \to \mathcal{C}$$

where we consider $\mathcal{B}$ as a full subcategory of $\mathrm{Open}(X)$*. It is called a sheaf whenever it satisfies the sheaf condition for all $U \in \mathcal{B}$ and all open covers by elements in $\mathcal{B}$.*

LEMMA 8.10. *A sheaf on $\mathcal{B}$ extends essentially uniquely to a sheaf on X. More precisely the forgetful functor from sheaves on X to sheaves on B*

$$U : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{B}}(X; \mathcal{C})$$

which simply forgets the value on all opens not contained in $\mathcal{B}$ is an equivalence of categories.

PROOF. We construct an inverse functor

$$\mathrm{Shv}_{\mathcal{B}}(X; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C}) \qquad F \mapsto \overline{F}.$$

We simply define $\overline{F}(U) := \lim_{V \subseteq U, V \in \mathcal{B}} F(V)$ where the limit is taken over the opposite of the category associated with the poset of all subsets $V \subseteq U$ contained in $\mathcal{B}$. This construction defines a functor $\mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ in the following way: for a given inclusion $U \subseteq U'$ we can consider elements in the poset $\{V \subseteq U \mid V \in \mathcal{B}\}$ also as elements of $\{V \subseteq U' \mid V \in \mathcal{B}\}$ and therefore get a 'projection'

$$\lim_{V \subseteq U', V \in \mathcal{B}} F(V) \to \lim_{V \subseteq U, V \in \mathcal{B}} F(V) \ .$$

This clearly defines a functor $\overline{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$. We have to verify that this is a sheaf. This will be left as an exercise!
Now for an open U in $\mathcal{B}$ we have

$$\overline{F}(U) = F(U)$$

because in this case the poset $\{V \subseteq U \mid V \in \mathcal{B}\}$ has a terminal object (namely U) so that the limit is simply evaluation at this object. We see that $U\overline{F}$ is naturally equivalent to F (the fact that all these constructions are naturaly is easily verified). Conversely if we start with a sheaf F on X we get a natural map

$$F \to \overline{UF}$$

given on an open set $U \subseteq X$ by the morphism

$$F(U) \to \lim_{V \subseteq U, V \in \mathcal{B}} F(V) \tag{4}$$

induced by the restrictions $F(U) \to F(V)$. This is an isomorphism as seen before. $\square$

THEOREM 8.11. *There is a (unique) sheaf $\mathcal{O}$ of commutative rings on* $\mathrm{Spec}(A)$ *which takes a principal open $U = D_f$ to*

$$\mathcal{O}(U) = A[1/f]$$

and an inclusion $U = D_f \subseteq D_g = V$ to the localization

$$\mathcal{O}(V) = A[1/g] \to A[1/f] = \mathcal{O}(U) \ .$$

PROOF. First we note that this makes sense, since for $D_f \subseteq D_g$ we have that $\sqrt{(f)} \subseteq \sqrt{(g)}$, thus $f^n = hg$ for some n and h. Thus inverting f also inverts g and we get a unique map $A[1/g] \to A[1/f]$. From this uniqueness also follows functoriality, i.e.we get a functor

$$\mathcal{O} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Ring} \ .$$

Thus in order to apply the previous lemma we need to verify that $\mathcal{O}$ is a sheaf on $\mathcal{B}$.

We need to verify the sheaf condition for $U = D_f$ and $U = \bigcup_i D_{f_i}$. Since U itself is given by $\mathrm{Spec}(A')$ with $A' = A[1/f]$ we can without loss of generality assume $f = 1$, i.e. that $\mathrm{Spec}(A) = \bigcup_i D_{f_i}$. We will work under the additional assumption that I is finite. The next lemma will show that this is indeed enough.
We have to show that

$$A \longrightarrow \prod_{i\in I} A[1/f_i] \Longrightarrow \prod_{i,j\in I} A[1/(f_i f_j)]$$

is an equalizer. The open set D_{f_i} does not change if we replace f_i by a power $f_i^{n_i}$. Then the fact that $\bigcup D_{f_i^{n_i}} = \mathrm{Spec}(A)$ shows that $A = \sqrt{(f_i^{n_i})}$, i.e. some power of 1 lies in $(f_i^{n_i})$, i.e. 1 does. In particular, we have that for every sequence of natural numbers $(n_i)_{i\in I}$ that $1 \in A$ can be written as a sum

$$1 = \sum_I g_i f_i^{n_i} \tag{5}$$

for elements $g_i \in A$.

Assume now that $s \in A$ lies in the kernel of the map

$$A \to \prod_{i\in I} A[1/f_i] \ .$$

Then we know that there exist n_i such that $f_i^{n_i} s = 0$. But then choosing the g_i as in (5) we get that

$$s = 1 \cdot s = \sum_I g_i f_i^{n_i} s = 0 \ .$$

This shows that the map from A into the equalizer is injective. For surjectivity assume that we have elements

$$s_i = a_i / f_i^{n_i} \in A[1/f_i]$$

such that the images of $a_i/f_i^{n_i}$ and $a_j/f_j^{n_j}$ agree in $A[1/(f_i f_j)]$, that is

$$a_i f_j^{n_j} f_i^m f_j^m = a_j f_i^{n_i} f_i^m f_j^m$$

in A, for some $m = m(i,j)$. In particular in $A[1/f_j]$ we have

$$(a_i f_i^m) f_j^{n_j} = a_j f_i^{n_i + m}$$

We replace a_i by $a_i f_i^m$ and n_i by $n_i + m$. Doing this for each i and j we can without loss of generality assume that we have $s_i = a_i / f_i^{n_i}$ and in $A[1/f_j]$ we have

$$a_i f_j^{n_j} = a_j f_i^{n_i} \tag{6}$$

Now we again choose g_i as above and set

$$s := \sum g_i a_i \ .$$

Now in $A[1/f_j]$ we have that

$$f_j^{n_j} s = \sum_i g_i f_j^{n_j} a_i = \sum g_i f_i^{n_i} a_j = a_j,$$

thus $s = s_j$ in $A[1/f_j]$ which finishes the proof modulo the next Lemma. □

LEMMA 8.12. *A presheaf* $F \in \mathrm{PSh}_{\mathcal{B}}(X; \mathrm{Ring})$ *of rings on principal opens in* $\mathrm{Spec}(A)$ *that satisfied the sheaf condition for all finite open covers* $\{U_i\}_{i\in I}$ *of a principal open* U *is already a sheaf.*

Proof. We want to show that F is separated first, that is for an arbirary cover $\{U_i\}_{i\in I}$ of U (not necessarily finite) the map

$$F(U) \to \prod_I F(U_i) \tag{7}$$

is injective. To see this we observe that U is quasi-compact, thus we have a finite subcover $\{U_i\}_{i\in I_0}$ that covers U. But then the composition

$$F(U) \to \prod_I F(U_i) \to \prod_{I_0} F(U_i)$$

is injective, thus also the map (7).
Now we want to show that for a given element s_i in the equalizer

$$\textstyle\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

we find an s. Again we pick a finite subcover and observe that our element $(s_i)_{i\in I}$ gives rise to an element $(s_i)_{i\in I_0}$. By the sheaf condition for finite covers we conlude that there is some $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I_0$. We want to show that we have this for all $i \in I$. Thus for a fixed i we observe that the restriction of s_i to $U_i \cap U_{i_0} \subseteq U_{i_0}$ agrees with the restriction of $s|_{U_i}$, since we have

$$(s|_{U_i})|_{U_i\cap U_{i_0}} = (s|_{U_{i_0}})|_{U_i\cap U_{i_0}} = (s_{i_0})|_{U_i\cap U_{i_0}} = (s_i)|_{U_i\cap U_{i_0}} \ .$$

This holds for all i_0. But the sets $\{U_i \cap U_{i_0}\}_{i\in I_0}$ for varying i_0 cover U_i, so that it follows from the separatedness that the elements s_i and $s|_{U_i}$ actually agree. (Note that here we are using that the U_i themselves are quasicompact, since they are principal opens and thus themselves of the form $\mathrm{Spec}(A')$.) □

Example 8.13. Consider the space $\mathrm{Spec}(\mathbb{Z})$. The open sets are precisely the sets $D(n)$ for $n \in \mathrm{N}$ consisting of all the prime divisors of some given number n. The structure sheaf $\mathcal{O}$ evaluated on this is given by

$$\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}(D(n)) = \mathbb{Z}[1/n].$$

We have an inclusion $D(n) \subseteq D(m)$ precisely if m divides n and then we get the restriction

$$\mathbb{Z}[1/m] \to \mathbb{Z}[1/n] \ .$$

9. Locally ringed spaces

From now on we shall assume that our category $\mathcal{C}$ is given by any of the categories Set, Ring, Ab, ...

Definition 9.1. *Let $F \in \mathrm{PSh}(X;\mathcal{C})$ be a presheaf and $x \in X$ a point. We define the* stalk *of F at x to be the colimit*

$$F_x := \varinjlim\nolimits_{U\in \mathrm{Open}(X), x\in U} F(U) \ .$$

For a given element $s \in F(U)$ with $x \in U$ we say that the image of s in F_x is the germ of s.

Note that this is a filtered colimit since the diagram is filtered: for any pair of open sets U, V with $x \in U, V$ we can take the intersection $U \cap V$ which still contains x and is contained in both of them.
Concretely an element of F_x is given by an equivalence class of pairs (U, f) where U is open containing x and $f \in F(U)$. Two such pairs (U, f) and (V, g) are declared

equivalent if there exists an open $W \subseteq U \cap V$ containing x such that $f|W = g|W$. So informally we think of elements as germs of 'functions around x'.

EXAMPLE 9.2. For $F = C^0(-, \mathbb{R})$ we have that we can really think of the stalk F_x as continuous functions defined on a very small neighborhood of x (and identified if they agree even closer to x).

REMARK 9.3. For the case of a basis of the topology $\mathcal{B}$ we have that we can also write the stalk as

$$F_x := \underrightarrow{\operatorname{colim}}_{U \in \mathcal{B}, x \in U} F(U) \ .$$

This follows simply since each open neighborhood of U contains a smaller one in $\mathcal{B}$.

EXAMPLE 9.4. Lets consider the case of the structure sheaf $\mathcal{O}$ on $\operatorname{Spec}(A)$. Then the stalk at $x \in \operatorname{Spec}(A)$ is given by

$$\mathcal{O}_x = \operatorname{colim}_{x \in D_f} A[1/f] \ .$$

Here we have that $x \in D_f$ translates into $f \notin x$. The structure maps of the colimit are the maps $A[1/f] \to A[1/g]$ for $D_g \subseteq D_f$ (i.e. $\sqrt{(g)} \subseteq \sqrt{(f)}$). We claim that this ring is the localization

$$A[1/f \mid f \notin x] =: A_x,$$

i.e. the universal ring obtained from A by inverting all elements in $A \setminus x$. Note that we can form the ring A_x for every prime ideal $x \subseteq A$ and it is called the *localization of A at x*.
This abstract ring has a universal property, namely that the elements in $A \setminus x$ are mapped to units in this ring and for any other ring B we have that the induced map

$$\operatorname{Hom}_{\mathrm{Ring}}(A_x, B) \to \operatorname{Hom}_{\mathrm{Ring}}(A, B)$$

obtained by composition with $A \to A_x$ is injective with image those morphisms $A \to B$ that send $A \setminus x$ to units. Thus in order to prove the isomorphism

$$\mathcal{O}_{\operatorname{Spec}(A),x} \cong A_x$$

we have to verify this universal property for $\operatorname{colim}_{x \in D_f} A[1/f]$: first of all, every element $f \in A$ that does not lie in x gets mapped to a unit in this colimit since it gets mapped to a unit in a finite stage. By definition with have that a morphism from this colimit to B is given by compatible maps $A[1/f] \to B$ for all D_f. Thus a morphism from this colimit to B consists exactly of a single morphism $A \to B$ such that all f get mapped to units.

DEFINITION 9.5. *A ring A is called a* local ring *if it has a unique maximal ideal $\mathfrak{m} \subseteq A$.*

EXAMPLE 9.6. Fields are local rings with the zero ideal being maximal. The integers $\mathbb{Z}$ are not a local ring, since there are many different maximal ideals. Similarly the rings $k[x_1, ..., x_n]$ are not local.

LEMMA 9.7. *For a ring A the following are equivalent:*

(1) *A is local*
(2) *The non-units $A \setminus A^\times$ form an ideal*
(3) *There exists an ideal $\mathfrak{a}$ such that $A \setminus \mathfrak{a}$ consists of units*
(4) $\operatorname{Spec}(A)$ *has a unique closed point.*

Proof. (1) $\Rightarrow$ (2): assume A is local. We claim that $\mathfrak{m} = A \setminus A^\times$. The inclusion $\subseteq$ is clear, since no maximal ideal can contain a unit. Conversely assume that $a \in A$ is not a unit. Then $(a) \neq A$ and (a) is contained in a maximal ideal, hence in $\mathfrak{m}$.
(2) $\Rightarrow$ (3): Take $\mathfrak{a} = A \setminus A^\times$.
(3) $\Rightarrow$ (1): We have to show that any ideal $I \subsetneq A$ is contained in $\mathfrak{a}$. Thus given an $i \in I$, then i is not a unit. Thus i is not contained in $A \setminus \mathfrak{a}$, hence in $\mathfrak{a}$.
The equivalence of (1) and (4) is immediate, since the closed points of $\mathrm{Spec}(A)$ are the closed ideals. $\square$

Example 9.8. Consider for a topological space X with the sheaf $C^0_X \in \mathrm{Shv}(X; \mathrm{Ring})$ of continuous $\mathbb{R}$-valued functions on X the stalk $C^0_{X,x}$. This is a local ring. An element $[f : U \to \mathbb{R}]$ is a unit precisely if $f(x) \neq 0$ since we can then find a neighborhood on which f is not zero and thus invert it. The complement consists of all f that vanish at x and is clearly an ideal.

Recall that for a prime ideal $x \subseteq A$ in a ring we define the localization of A at x denoted A_x by inverting all elements in $A \setminus x$.

Proposition 9.9. *For every prime ideal $x \in \mathrm{Spec}(A)$ the stalk $\mathcal{O}_{\mathrm{Spec}(A),x} = A_x$ is a local ring. The map*

$$\mathrm{Spec}(A_x) \to \mathrm{Spec}(A)$$

exhibits $\mathrm{Spec}(A_x)$ as a subspace given by the intersection of all opens $U \subseteq \mathrm{Spec}(A)$ that contain x.

Proof. Generally, the prime ideals in a ring $A[S^{-1}]$ for some set $S \subseteq A$ are given by the prime ideals of A that do not contain S, which follows as in Proposition 6.14. We conclude that in our case the prime ideals of A_x are in 1-1 correspondence to prime ideals of A that are contained in x. It therefore immediately follows that there is a unique maximal ideal (namely x itself) and that A_x is local. It also follows that

$$\mathrm{Spec}(A_x) = \{y \in \mathrm{Spec}(A) \mid y \subseteq x\}$$

is a subset and even a subspace of $\mathrm{Spec}(A)$ similarly to the proof of Proposition 6.14. Again this is a general statement about inverting a subset $S \subseteq A$.

Thus it only remains to identify this subset with the subset

$$\bigcap_{M \subsetneq x} D(M) = \bigcap_{M \subsetneq x} \{y \in \mathrm{Spec}(A) \mid M \subsetneq y\} = \bigcap_{m \notin x} \{y \in \mathrm{Spec}(A) \mid m \notin y\}$$

which is clear. $\square$

Note that in particular the rings $\mathbb{Z}_{(p)}$ are local. These can canonically be considered as subsets of $\mathbb{Q}$. The maximal ideal is then given by (p).

Definition 9.10. *A* ringed space *is a pair $(X, \mathcal{O}_X)$ of a topological space and $\mathcal{O}_X$ a sheaf of rings. It is called locally ringed space if all the stalks $\mathcal{O}_{X,x}$ are local rings.*

Example 9.11. For any ring A the pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space. We will also simply denote this pair as $\mathrm{Spec}(A)$ and leave the sheaf of rings implicit (so be careful if we mean $\mathrm{Spec}(A)$ as a plain topological space or a locally ringed space).
For any topological space X we have the locally ringed space (X, C^0_X).

EXAMPLE 9.12. The pair (X, Abb), where Abb denotes the sheaf $\mathrm{Hom}_{\mathrm{Set}}(-;\mathbb{R})$, is not a locally ringed space unless X carries the discrete topology (in which case it agrees wth (X, C^0)). To see this assume that X does not carry the discrete topology and pick a point $x \in X$ for which every open U containing x has at least one other point. Then we consider the germs of the two functions

$$f_1(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{else} \end{cases} \qquad f_2(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{else} \end{cases}$$

These two germs $f_1, f_2 \in \mathrm{Abb}_x$ are not units (being a unit means that a function has to be non-zero in a neighborhood of x). But their sum is a unit, since it is constant 1. Therefore the ring cannot be local, by criterion (2) of Lemma 9.7.
Given any ring R which is not local, we can also consider the pair (pt, R) where we consider R as a sheaf on the one point space pt. Concretely we have that this sheaf sends $\emptyset$ to the zero-ring (the terminal object in rings) and the full subset pt to R. Then the stalk of this sheaf is simply R which is not local.

DEFINITION 9.13. *For a locally ringed space $(X, \mathcal{O}_X)$ and $x \in X$ we define the residue field $\kappa(x)$ at x as the quotient*

$$\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$$

where $\mathfrak{m}_x$ is the maximal ideal. For any $f \in \mathcal{O}(U)$ and $x \in U$ we define $f(x)$ to be the class of $[U, f]$ in $\kappa(x)$.

EXAMPLE 9.14. For the locally ringed space (X, C^0_X) all the residue fields are $\mathbb{R}$ and for $f : U \to \mathbb{R}$ and $x \in U$ this simply evaluates at x.
For $\mathrm{Spec}(A)$ we have that

$$\kappa(x) = A_x/\mathfrak{m}_x = A[(A \setminus x)^{-1}]/\mathfrak{m}_x = (A/x)[(A \setminus x)^{-1}] = \mathrm{Quot}(A/\mathfrak{m}).$$

This was the definition of $\kappa(x)$ as given before (Definition 6.1). For $f \in A = \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$ we also find that under this identification the two definition of $f(x)$ agree.

10. Morphisms of locally ringed spaces

Before we can talk about morphisms now we need the following construction:

CONSTRUCTION 10.1. Let $f : X \to Y$ be a continuous map of topological spaces and $F \in \mathrm{PSh}(X; \mathcal{C})$. We construct a presheaf $f_*(F) \in \mathrm{PSh}(Y; \mathcal{C})$ by setting

$$f_*(F)(U) = F(f^{-1}(U)) \ .$$

This is a presheaf, i.e. functorial in U. This is clear, but a more formal way of saying this is to observe that f induces a functor $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ and we have simply composed with this functor:

$$f_*(F) : \mathrm{Open}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathrm{Open}(X)^{\mathrm{op}} \xrightarrow{F} \mathcal{C} \ .$$

This assignment defines a functor

$$f_* : \mathrm{PSh}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C}) \ .$$

LEMMA 10.2. *Assume that $F \in \mathrm{PSh}(X; \mathcal{C})$ is a sheaf. Then so is f_*F.*

Proof. Assume that we have an open covering U_i of U in Y. Then the pullback $f^{-1}U_i$ is an open covering of $f^{-1}U$ and we have that $f^{-1}U_i \cap f^{-1}U_j = f^{-1}(U_i \cap U_j)$. Therefore the sheaf condition for f_*F is simply given by the diagram

$$F(f^{-1}U) \longrightarrow \prod_{i\in I} F(f^{-1}U_i) \rightrightarrows \prod_{i,j\in I} F(f^{-1}U_i \cap f^{-1}U_j)$$

which is satisfied since F is a sheaf. $\square$

Definition 10.3. *A morphism of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is given by a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is a morphism of sheaves of rings over Y.*

Example 10.4. Let $f : X \to Y$ be a continuous map. Then we get an induced morphism

$$f^\sharp : C^0_Y \to f_*(C^0_X)$$

which on an open set $U \subseteq Y$ is given by the map

$$C^0(U; \mathbb{R}) \to C^0(f^{-1}U; \mathbb{R})$$

which composes a map $\varphi : U \to \mathbb{R}$ with the map $f : f^{-1}U \to U$ to get a map $f^\sharp(\varphi) : f^{-1}U \to \mathbb{R}$. This is a morphism of sheaves as one directly sees. Thus the morphisms $f^\sharp$ should be seen as the pullback of functions.

Example 10.5. Let $f : X \to Y$ be a smooth map between smooth manifolds. Then we get an induced morphism

$$f^\sharp : C^\infty_Y \to f_*(C^\infty_X)$$

which on an open set $U \subseteq Y$ is given by the composition map

$$C^\infty(U; \mathbb{R}) \to C^\infty(f^{-1}(U); \mathbb{R})$$

as before. In fact, one can show that for a continuous map $f : X \to Y$ the induced morphism

$$f^\sharp : C^0_Y \to f_*(C^0_X)$$

restricts to a morphism $C^\infty_Y \to f_*(C^\infty_X)$ precisely if f is smooth. In other words: a continuous map $f : X \to Y$ between smooth manifolds X and Y is smooth precisely of for each smooth function $\varphi : U \to \mathbb{R}$ with $U \subseteq Y$ the composition $\varphi \circ f : f^{-1}(U) \to U \to \mathbb{R}$ is smooth.

Example 10.6. Let $f : A \to B$ be a morphism of rings. Then we claim that we get an induced morphism of ringed spaces

$$\operatorname{Spec}(f) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

which on underlying topological spaces is given by the morphism $\operatorname{Spec}(f)$ defined before (see Proposition 6.7) and which simply takes preimages of prime ideals. Now we want to extend this to a morphism of ringed spaces, i.e. we have to define

$$\operatorname{Spec}(f)^\sharp : \mathcal{O}_{\operatorname{Spec}(A)} \to \operatorname{Spec}(f)_*\mathcal{O}_{\operatorname{Spec}(B)} .$$

By the equivalence of Lemma 8.10 it suffices to define such a morphism on principal opens, i.e. for each principal open $D_g \subseteq \operatorname{Spec}(A)$ with $g \in A$ a morphism

$$\mathcal{O}_{\operatorname{Spec}(A)}(D_g) \to \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(f)^{-1}(D_g))$$

natural in the open set. Note that $\operatorname{Spec}(f)^{-1}(D_g) = D_{f(g)}$, as we proved when proving continuity of $\operatorname{Spec}(f)$. So we need to give a map

$$A[1/g] \to B[1/f(g)]$$

which is simply the map induced by f. Naturality is then straightforward.

Now we would like to define morphisms of locally ringed spaces. To this end we first have to talk about morphisms of local rings. For every ring morphisms $f : A \to B$ between local rings we clearly have that $f^{-1}(\mathfrak{m}_B)$ is an ideal in B. Since $f^{-1}(\mathfrak{m}_B) \neq A$ we have that $f^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. In general there is no equality as the example of the map

$$\mathbb{Z}_{(p)} \to \mathbb{Q}$$

shows. The preimage of 0 is given by 0 which is not the maximal ideal of $\mathbb{Z}_{(p)}$.

Definition 10.7. *A morphism of local rings is a morphism $f : A \to B$ such that $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.*

Remark 10.8. Note that the definition is chosen in such a way that morphisms of local rings $f : A \to B$ induce maps between residue fields $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$. (It is equivalent to $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.) In fact, this gives us a functor from the category of local rings to the category of fields. There is no such functor on the full subcategory of Ring on the local rings.

Lemma 10.9. *A ring homomorphism between local rings is a morphism of local rings iff the induced map* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *of topological spaces maps the closed point to the closed point.*

Proof. Clear. □

Example 10.10. Every isomorphism $A \to B$ between local rings is a morphism of local rings.

Construction 10.11. Now let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For every point $x \in X$ there is a canonical morphism of rings

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

induced by $(f, f^\sharp)$. This morphism sends (U, φ) to $(f^{-1}U, f^\sharp\varphi)$. By this requirement the morphism is completely determined and it is in fact easy to verify the claim here directly (one simply needs to check well-definedness, but by assumption $f^\sharp$ is compatible with restriction, so this is clear). But we will say things a bit more systematically in the next section.

Definition 10.12. *A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of the underlying ringed spaces such that the induced morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a morphism of local rings.*

Example 10.13. For a continuous map $f : X \to Y$ of topological spaces the induced morphism $(f, f^\sharp) : (X, C^0_X) \to (Y, C^0_Y)$ as in Example 10.4 is on germs the induced map

$$C^0_{Y,f(x)} \to C^0_{X,x}$$

sends a germ $f : U \to \mathbb{R}$ to the composition $f^{-1}U \to U \to \mathbb{R}$. If f is in the maximal ideal (i.e. vanishes at $f(x)$) then this composition lies also in the maximal ideal. Therefore this is a map of locally ringed spaces.

Similar we claim that $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for $f : A \to B$ is a morphism of locally ringed space. Under the appropriate definitions the induced morphism on stalks is simply given by the map

$$f : A_{f^{-1}(x)} \to B_x$$

for $x \in \mathrm{Spec}(B)$. Clearly, this takes the maximal ideal $(f^{-1}(x))$ into the maximal ideal (x).

REMARK 10.14. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of locally ringed spaces, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces a map $\kappa(f(x)) \to \kappa(x)$ of fields. If we think of $f^\sharp$ as precomposing a function $g \in \mathcal{O}_Y(U)$ with f, then $\kappa(f(x)) \to \kappa(x)$ tells us how the value $g(f(x)) \in \kappa(f(x))$ relates to the value of the restriction $f^\sharp(g)(x) \in \kappa(x)$.
For a map $f : X \to Y$ of topological spaces considered as a map of locally ringed spaces $(X, C^0_X) \to (Y, C^0_Y)$ the maps $\kappa(f(x)) \to \kappa(x)$ are all the identity $\mathbb{R} \to \mathbb{R}$, so in that case this simply expresses that $(g \circ f)(x) = g(f(x))$.

THEOREM 10.15. *The functor* $\mathrm{Spec}(-)$ *from commutative rings to locally ringed spaces is fully faithful.*

We will prove this later, in fact deduce it from a more general statement.

11. Adjunctions between sheaves

LEMMA 11.1. *A morphism of sheaves $F \to G$ with values in rings/sets/abelian groups on a space X is an isomorphism, precisely if it is an isomorphism on stalks.*

PROOF. The only if statement is clear. Thus assume that $F \to G$ induces an isomorphism on stalks.
Observe that the map $F(U) \to G(U)$ is injective for each U: If $f, g \in F(U)$ have the same image in $G(U)$, they in particular have the same image in the stalk G_x for each $x \in U$. Since the map on stalks $F_x \to G_x$ is injective, this shows that f, g have the same image in F_x for each $x \in U$. By the explicit description of stalks, this shows that for each $x \in U$, there is a neighbourhood $U_x \subseteq U$ such that f, g agree already in $F(U_x)$. The U_x cover U, so f, g agree in $F(U)$.
For surjectivity, pick some $f \in G(U)$. For each $x \in U$, we find some element of F_x which maps to the same element as f in G_x. Representing this by some $f_x \in F(U_x)$ for a neighbourhood, we see that the image of f_x in $G(U_x)$ has the same germ at x as f. Thus, we can make U_x smaller to assume that the image of f_x in $G(U_x)$ agrees with $f|_{U_x}$.
What we have achieved is that we have found an open cover U_x of U and preimages $f_x \in F(U_x)$ of $f|_{U_x} \in G(U_x)$. In particular, the images of f_x and f_y in $G(U_x \cap U_y)$ agree. By injectivity this shows that $f_x|_{U_x \cap U_y} = f_y|_{U_x \cap U_y}$. It follows that the f_x glue to an element of $F(U)$, whose image in $G(U)$ agrees with f since it does on the cover U_x. □

REMARK 11.2. Note that the injectivity portion of the proof only required injectivity on stalks, i.e. if $F_x \to G_x$ is injective for all $x \in U$, $F(U) \to G(U)$ is, too.
The corresponding statement for surjectivity does not hold! As an example, consider $X = \mathbb{C} \setminus \{0\}$, with sheaf $C^0(-; \mathbb{C})$ of continuous complex-valued functions. We have the squaring map

$$C^0(U; \mathbb{C}) \to C^0(U; \mathbb{C}), \quad f \mapsto f^2,$$

which is surjective on stalks, but not on $\mathbb{C}\setminus\{0\}$. (Otherwise, we would have a global square root map as a preimage of the "identity" $U \to \mathbb{C}$.)

So isomorphisms of sheaves are detected on stalks. As we will now see, there is also a universal way of turning a presheaf into a sheaf without changing the stalks.

THEOREM 11.3. *For every presheaf F of rings/sets/abelian groups there exists a universal sheaf F' with a morphism $F \to F'$, i.e. such that every other morphism $F \to G$ with G a sheaf factors through F'. The morphism $F \to F'$ induces an isomorphism on stalks, and F' is characterized by this property.*

PROOF. We set

$$F'(U) \subseteq \{(f_x \in F_x)_{x \in U}\}$$

where an element lies in $F'(U)$ if each x admits a neighbourhood U_x and $g \in F(U_x)$ such that for all $y \in U_x$ the germ of g agrees with f_y. So $F'(U)$ consist of "functions on U with values in the stalks" with the property that they locally come from actual sections of F.
This is functorial in the obvious way. We get a presheaf map $F \to F'$ which takes an actual element $f \in F(U)$ to the collection of its germs. On stalks, this map is surjective, since by definition every element of $F'(U)$ with $x \in U$ lifts to $F(V)$ on some smaller neighbourhood V of x. It is also injective, since if two $f, g \in F_x$ have the same image in F'_x, their representatives coincide in $F'(V)$ for some neighbourhood V of x, so in particular in F_x.
We now directly check that F' is indeed a sheaf: A section on U consist of compatible sections on all U_i for some cover, since both amounts to give an element $f_x \in F_x$ for each $x \in U$, with the property that this assignment locally comes from actual sections in F. But since this last condition is local anyways, it can be checked on the cover.
We now check the universal property. Given another map $F \to G$ with G a sheaf, we get a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "\cong"] \\ F' \arrow[r] \arrow[ur, dashed] & G' \end{tikzcd}$$

The lower horizontal map is uniquely determined by how $F \to G$ acts on stalks. The right vertical map is an isomorphism of sheaves by the preceding lemma. So there exists a unique dashed morphism of sheaves making the upper left triangle commute.
The final claim is that F' is also characterized simply by being a sheaf with a map $F \to F'$ which is an isomorphism on stalks. Given another morphism $F \to G$ which is an isomorphism on stalks, with G a sheaf, by the universal property we get a morphism $F' \to G$ of sheaves which also is an isomorphism on stalks. So it is an isomorphism, again by the preceding lemma.

□

REMARK 11.4. The theorem is true, with a different proof, way more generally for a large class if categories $\mathcal{C}$ (essentially, $\mathcal{C}$ needs all colimits and limits and a certain interchange property that is practically always satisfied).

REMARK 11.5. The assignment $F \mapsto F'$ defines a functor $\mathrm{PSh}(X;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$ by the universal property. This is the *sheafification functor a*, we will therefore write F' as aF usually.

DEFINITION 11.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An* adjunction *between a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ consist of an equivalence*

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy)$$

natural in x and y (i.e. we think of both sides as functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$.)
In this situation we call F the left adjoint and G the right adjoint, and say things like "G is right adjoint to F". We write $F \dashv G$ to express the relationship.

EXAMPLE 11.7. Sheafification is left adjoint to the forgetful functor $i : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{PSh}(X; \mathcal{C})$, i.e. $a \dashv i$, so for each pair $F \in \mathrm{PSh}(X; \mathcal{C})$, $G \in \mathrm{Shv}(X; \mathcal{C})$ we have

$$\mathrm{Hom}_{\mathrm{PSh}(X;\mathcal{C})}(F, iG) \cong \mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(aF, G).$$

This is precisely the universal property of sheafification.

EXAMPLE 11.8. Sobrification is left adjoint to the forgetful functor from sober spaces to spaces.

EXAMPLE 11.9. The functor taking a local ring to its residue field is left adjoint to the forgetful functor from fields to the category of local rings. This is because we have

$$\mathrm{Hom}_{\mathrm{locRing}}(R, k) \cong \mathrm{Hom}_{\mathrm{field}}(R/\mathfrak{m}, k)$$

EXAMPLE 11.10. The forgetful functor Ring $\to$ Set has a left adjoint. It is given by the functor which takes $X \in \mathrm{Set}$ to the polynomial ring $\mathbb{Z}[X] = \mathbb{Z}[x_i \mid i \in X]$. This is the universal property of polynomial rings. (Similarly, the functor taking a set to the free group, free abelian group, free K-vector space,... can be described as a left adjoint to a forgetful functor.)

We now recall the Yoneda Lemma. This is about the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (sometimes also refered to presheafs on $\mathcal{C}$ since it is the obvious generalization). Every object $c \in \mathcal{C}$ gives rise to such a functor denoted $\underline{c}$ as follows:

$$\underline{c} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \qquad x \mapsto \mathrm{Hom}_{\mathcal{C}}(x, c)\ .$$

PROPOSITION 11.11 (Yoneda Lemma). *For $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and $c \in \mathcal{C}$ there is a (natural) isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})}(\underline{c}, F) \cong F(c)\ .$$

PROOF. We explicitly give a map $\phi : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})}(\underline{c}, F) \to F(c)$: for a natural transformation $\eta : \underline{c} \to F$ we let $\phi(\eta)$ be the value

$$\mathrm{id}_c \in \underline{c}(c) \xrightarrow{\eta} F(c)\ .$$

Conversely for a given $d \in F(c)$ we consider the natural transformation $\eta_d : \underline{c} \to F$ which sends $f : d \to c$ to the element in $F(d)$ given by the image under

$$F(c) \xrightarrow{f} F(d)\ .$$

One readily checks that these two maps are inverse to each other. □

COROLLARY 11.12. *The Yoneda embedding*

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \qquad c \mapsto \underline{c}$$

is fully faithful. In particular if for two objects $c, d \in \mathcal{C}$ the functors $\underline{c}, \underline{d}$ are naturally equivalent, then c and d are isomorphic.

We say that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable if it is in the essential image of the Yoneda embedding, that is if it is isomorphic to $\underline{c}$ for some c. This object is then uniquely determined.
Of course we also have the dual version applying this to $\mathcal{C}^{\mathrm{op}}$ which says that $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ sending $c \in \mathcal{C}$ to the functor $d \mapsto \mathrm{Hom}_{\mathcal{C}}(c, d)$. This is also fully faithful and thus we conclude that mapping out of an object also uniquely characterises objects of categories.

LEMMA 11.13. *If a functor has a right adjoint, it is unique up to canonical natural isomorphism, and dually for left adjoints.*

PROOF. Assume G and G' are two right adjoints to $F : \mathcal{C} \to \mathcal{D}$. We get a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \cong \mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G'y),$$

which, by the Yoneda lemma, gives rise to a natural isomorphism $Gy \to G'y$ (we get the map by looking at what happens to $\mathrm{id}_{Gy} \in \mathrm{Hom}_{\mathcal{C}}(Gy, Gy)$ on the left, checking that it gives an isomorphism is a little diagram chase). $\square$

LEMMA 11.14. *Right adjoint functors preserve limits. Left adjoint functors preserve colimits.*

PROOF. We again show one direction. Assume G is right adjoint to $F : \mathcal{C} \to \mathcal{D}$. Let $y \to (y_i)_{i \in I}$ be some limit cone over a diagram $I \to \mathcal{D}$. We want to show that $Gy \to (Gy_i)_{i \in I}$ is a limit cone. This means we have to show that given any cone $x \to (Gy_i)_{i \in I}$, it factors uniquely through Gy. By adjoining, the maps in the cone give a cone $Fx \to (y_i)_{i \in I}$ in $\mathcal{D}$. By assumption, this factors through a map $Fx \to y$. The adjoint map $x \to Gy$ factors the given cone. $\square$

For example, this explains why limits of rings are formed "underlying", i.e. the limit in the category of rings is a ring whose underlying set is just the limit of the sets. This is since the forgetful functor $\mathrm{Ring} \to \mathrm{Set}$ has a left adjoint. The same for colimits is of course not true (and the forgetful functor does not have a right adjoint).
This also explains why a limit of sheaves is again a sheaf, since the fully faithful functor $i : \mathrm{Shv} \to \mathrm{PSh}$ is a right adjoint. The same is typically not true for colimits.

LEMMA 11.15. *Given functors $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ which both admit right adjoints G_1 and G_2, $(G_1 \circ G_2)$ is right adjoint to $(F_2 \circ F_1)$. (Analogously for left adjoints).*

PROOF. We have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_3}(F_2F_1(x), y) = \mathrm{Hom}_{\mathcal{C}_2}(F_1(x), G_2(y)) = \mathrm{Hom}_{\mathcal{C}_1}(x, G_1G_2(y))$$

which shows the claim. $\square$

We will now turn back to the study of sheaves on spaces. Recall that we described for every continuous map $f : X \to Y$ the direct image functor $f_* : \mathrm{PSh}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C})$ which took sheaves to sheaves, i.e. we also have $f_* : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$.

LEMMA 11.16. *The functor $f_* : \mathrm{PSh}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C})$ has a left adjoint f^+, which takes F to $F^+ = f^+(F)$ with*

$$f^+(F)(U) = \mathrm{colim}_{V \supseteq f(U)} F(V).$$

PROOF. For each F, the construction $F^+ = f^+(F)$ defines a presheaf, with restriction map coming from the observation that if $U' \subseteq U$, then every $V \supseteq f(U)$ also satisfies $V \supseteq f(U')$, so we can map the colimits to each other. It is also clearly functorial in F. To see that it is left adjoint, we need to check that a map of presheaves $F^+ \to G$ is the same as a map $F \to f_*G$. Unwrapping the definitions, a map $F^+ \to G$ consists of maps

$$\operatorname{colim}_{V \supseteq f(U)} F(V) \to G(U),$$

i.e. of compatible maps $F(V) \to G(U)$ for each pair of $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, or $U \subseteq f^{-1}(V)$. By the same logic, such a collection of compatible maps also is the same as a map

$$F(V) \to \lim_{U \subseteq f^{-1}(V)} G(U) = G(f^{-1}V) = f_*G(V),$$

as desired. □

REMARK 11.17. Note that the colimit in the definition of $F^+(U)$ is filtered so that even in rings it is formed underlying.

This construction $F \mapsto f^+(F)$ does *not* preserve sheaves. Instead, we can get a similar adjoint also on the level of sheaves, by postcomposing with sheafification. This is the functor we are really after:

THEOREM 11.18. *The direct image functor $f_* : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$ admits a left adjoint f^{-1}. It can be explicitly described as taking F to the sheafification of*

$$U \mapsto \operatorname{colim}_{V \supseteq f(U)} F(V).$$

PROOF. We first observe that $f_* \circ i : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C})$ has a left adjoint $f^{-1} : \mathrm{PSh}(Y; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C})$, given by the functor from Lemma 11.16 followed by sheafification, by Lemma 11.15.
Explicitly, the adjunction gives a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(f^{-1}(F), G) \cong \mathrm{Hom}_{\mathrm{PSh}(Y;\mathcal{C})}(F, f_*G),$$

where we suppress i. We know that f_*G is also a sheaf, so if we consider the above natural isomorphism only for $F \in \mathrm{Shv}(Y; \mathcal{C})$, we also get a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(f^{-1}(F), G) \cong \mathrm{Hom}_{\mathrm{Shv}(Y;\mathcal{C})}(F, f_*G),$$

as desired. □

The inverse image functor is a common generalisation of restriction and of stalks.

EXAMPLE 11.19. For an open inclusion $i : U \to X$, the inverse image functor $\mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(U; \mathcal{C})$ simply restricts a sheaf: For each $V \in U$, we have

$$i^{-1}F(V) = \operatorname{colim}_{V' \supseteq i(V)} F(V') = F(i(V)),$$

since for open $i(V)$ the diagram has terminal object $i(V)$. A priori we need to sheafify this, but if i is an inclusion, this is already a sheaf. More generally this description works for any open map (not necessarily injective), but then we *do* need to sheafify! In this case we shall also sometimes simply write the functor $i^{-1}F$ as $F|_U$ and think of it simply as restriction of sheaves.

EXAMPLE 11.20. For the inclusion of a point $i_x : \mathrm{pt} \to Y$, $\mathrm{Shv}(\mathrm{pt}; \mathcal{C})$ is just $\mathcal{C}$, and the explicit formula for the inverse image functor tells us that $i_x^{-1}F$ is F_x under that identification.

EXAMPLE 11.21. We first note that the constant presheaf F on a space X with value R (a ring) is not a sheaf as we have seen in Example 8.6. But we can form the sheafification aF and we claim that this is given by locally constant functions on X with values in R, that is maps $X \to R$ for which every point $x \in X$ has a neighborhood U such that the function is constant in this neighboorhood. The latter is clearly a sheaf and the canonical map from constant into locally constant function is a morphism of sheaves which induces an isomorphism on stalks. Thus this is aF. We refer to this sheaf also as the *constant sheaf with value* R.

For the constant map $c : X \to \mathrm{pt}$, $c^{-1}R$ by definition is the sheafification of the presheaf

$$U \mapsto R,$$

i.e. the constant presheaf. Thus, $c^{-1}R$ is the "sheaf of locally constant functions with values in R". Note that the direct image also has a very important description: $c_*(F)$, regarded as object in $\mathcal{C}$, is given by the *global sections* $F(X)$. The adjunction thus gives us the important identity

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(c^{-1}R, F) \cong \mathrm{Hom}_{\mathcal{C}}(R, F(X)),$$

i.e. maps out of the sheaf of locally constant functions into any sheaf are uniquely determined by what they do on global sections.

LEMMA 11.22. *For $f : X \to Y$ we have for every x a natural isomorphism $(f^{-1}F)_x = F_{f(x)}$.*

PROOF. We have that for the composition $\mathrm{pt} \xrightarrow{i_x} X \xrightarrow{f} Y$ that $f_* \circ (i_x)_* = (f \circ i_x)_* = (i_{f(x)})_*$. Thus we get by Lemma 11.15 that $(i_x)^{-1} \circ f^{-1} = (i_{f(x)})^{-1}$ which by Example 11.20 gives the claim. $\square$

COROLLARY 11.23. *For a locally ringed space $(Y, \mathcal{O}_Y)$ and a continuous map $f : X \to Y$ the pair $(X, f^{-1}\mathcal{O}_Y)$ is also a locally ringed space. In particular for an open subset $U \subseteq Y$ the pair $(U, \mathcal{O}_Y|_U)$ is again a locally ringed space.*

EXAMPLE 11.24. Consider the open subset $D(f) \subseteq \mathrm{Spec}(A)$. Then this becomes a locally ringed space. We claim that as such it is isomorphic to $\mathrm{Spec}(A[1/f])$. We have shown this as topological spaces already in Proposition 6.14. Now we need to also compare the structure sheaves. But note that a principal open in $\mathrm{Spec}(A[1/f])$ is simply a principal open in $\mathrm{Spec}(A)$ of the form $D(g)$ such that $\sqrt{(g)} \subseteq \sqrt{(f)}$. But then

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(g)) = A[1/g] = A[1/f][1/g] = \mathcal{O}_{\mathrm{Spec}(A[1/f]}(D(g)) \ .$$

DEFINITION 11.25. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is called* open immersion *if it induces an isomorphism between X and $(U, \mathcal{O}_Y|_U)$ for some open $U \subseteq Y$.*

Note that a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ was defined as a pair $f : X \to Y$ and a morphism $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$. By adjunction we can now also think of $f^\sharp$ also as a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We will do that very often and also denote this map as $f^\sharp$ abusively. Then this morphism on stalks induces the map

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

that we have been using in the Definition of morphisms of locally ringed spaces (Definition 10.12). This we could rephrase this Definition as saying such a morphism is a pair $(f, f^\sharp)$ with $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that $f^\sharp$ is stalkwise a morphism of local rings.

12. Schemes and affinization

Recall that we stated as Theorem 10.15 that the functor $\mathrm{Spec}(-)$ from commutative rings to locally ringed spaces is fully faithful. In this Section we will prove this and a generalization thereof. We will deduce this from a more general result that we will describe now.

Definition 12.1. *The global sections functor* Γ *is the functor*

$$\Gamma : \{\textit{locally ringed spaces}\} \to \mathrm{Ring}^{\mathrm{op}}$$

which takes a locally ringed space $(X, \mathcal{O}_X)$ *to the global sections* $\mathcal{O}_X(X)$*. It sends a morphism* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *to the map* $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ *induced by* $f^\sharp$ *on* U*.*

Theorem 12.2. *The functor* $\Gamma : \{\textit{locally ringed spaces}\} \to \mathrm{Ring}^{\mathrm{op}}$ *is left adjoint to the functor* $\mathrm{Spec} : \mathrm{Ring}^{\mathrm{op}} \to \{\textit{locally ringed spaces}\}$*, that is we have a natural isomorphism*

$$\mathrm{Hom}\Big((X, \mathcal{O}_X), \mathrm{Spec}(A)\Big) \cong \mathrm{Hom}_{\mathrm{Ring}^{\mathrm{op}}}\Big(\mathcal{O}_X(X), A\Big) \cong \mathrm{Hom}_{\mathrm{Ring}}\Big(A, \mathcal{O}_X(X)\Big).$$

Let us first explain how this proves Theorem 10.15, we clearly immediately get an isomorphism

$$\mathrm{Hom}\Big(\mathrm{Spec}(B), \mathrm{Spec}(A)\Big) \cong \mathrm{Hom}_{\mathrm{Ring}}\Big(A, \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B))\Big) = \mathrm{Hom}_{\mathrm{Ring}}(A, B)$$

but this doesn't quite show the claim, since we need to show that this bijection is inverse to the map induced by $\mathrm{Spec}(-)$. In order to see this one has to go into the analysis of adjunctions a bit more. Let

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

be an adjunction. Then we get natural morphisms $\epsilon : LR(d) \to d$ as the image of the identity under the bijection

$$\mathrm{Hom}_{\mathcal{C}}(R(d), R(d)) \cong \mathrm{Hom}_{\mathcal{D}}(LR(d), d)$$

which is part of the adjunction. The naturality in $d \in \mathcal{D}$ is immediately verified. This transformation is called the *counit* of the adjunction. Dually there is also a *unit* $\eta : c \to RL(c)$ (but we focus on the counit for now). Now assume that we are given a transformation $\epsilon : LR \to \mathrm{id}$ as above, then we can define a natural map

$$\mathrm{Hom}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}(Lc, LRd) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{D}}(Lc, d)\ .$$

and if ϵ came from an adjunction we claim that this is indeed the natural isomorphism which is part of the adjunction. This follows from the Yoneda Lemma: for fixed d any morphism $\mathrm{Hom}_{\mathcal{C}}(-, Rd) \to \mathrm{Hom}_{\mathcal{D}}(L-, d)$ has to be induced by an element in $\mathrm{Hom}_{\mathcal{D}}(LRd, d)$, which is the counit and the way the morphism is induced is then the upper composition.

Remark 12.3. One can even revert the logic and give the following equivalent definition of an adjunction: it is given by a natural transformation $\epsilon : LR \to \mathrm{id}$ such that the induced natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(c, Rd) \to \mathrm{Hom}_{\mathcal{D}}(Lc, d)$$

is an isomorphism. Using this definition is sometimes benefical.

Proposition 12.4. *For an adjunction $L \dashv R$ the functor R is fully faithful precisely if the counit $LR \to \mathrm{id}$ is an isomorphism.*

Proof. We claim that the following triangle commutes

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(x,y) \arrow[r, "R"] \arrow[dr, "\epsilon^*"'] & \mathrm{Hom}_{\mathcal{C}}(Rx, Ry) \arrow[d, "\cong"] \\ & \mathrm{Hom}_{\mathcal{D}}(LR(x), y) \end{tikzcd}$$

This then immediately implies the claim since then the first morphisms is a bijection precisely if the latter one is, which by Yoneda is equivalent to the assertion that $LR(x) \to x$ is an isomorphism. To see the commutativity we consider the larger diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(x,y) \arrow[r, "R"] \arrow[d, "\mathrm{id}"] \arrow[dr, "LR"'] & \mathrm{Hom}_{\mathcal{C}}(Rx, Ry) \arrow[d, "L"] \\ \mathrm{Hom}_{\mathcal{D}}(x,y) \arrow[dr, "\epsilon_x"] & \mathrm{Hom}_{\mathcal{D}}(LRx, LRy) \arrow[d, "\epsilon_y"] \\ & \mathrm{Hom}_{\mathcal{D}}(LRx, y) \end{tikzcd}$$

where the commutativity of the lower square is the naturality of $\epsilon : LR \to \mathrm{id}$. $\square$

Now in order to deduce that $\mathrm{Spec}(-)$ is fully faithful we need to show that the counit of the adjunction is an isomorphism. In fact, we will show in the proof of Theorem 12.2 that the counit of the adjunction is given by the natural isomorphism $\Gamma(\mathrm{Spec}(A)) \cong A$. Before we give the proof we will give the main definition of the course, the definition of schemes. It is similar to that of a manifold, which is a topological space locally isomorphic to $\mathbb{R}^n$:

Definition 12.5. *An* affine scheme *is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to* $\mathrm{Spec}(A)$ *for some ring A. The category of affine schemes* AffSch *is the full subcategory of locally ringed spaces on affine schemes.*
A scheme *is a locally ringed space $(X, \mathcal{O}_X)$ which is locally affine, that is for which there exists an open cover U_i of X such that for each i the locally ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme (i.e. isomorphic to* $\mathrm{Spec}(A_i)$ *for some ring A_i) The category of schemes* Sch *is the full subcategory of locally ringed spaces on affine schemes.*

We will often write a scheme simply as X instead of $(X, \mathcal{O}_X)$ and leave the sheaf implicit.

Corollary 12.6. (1) *The functors* Spec *and* Γ *induce inverse equivalences*

$$\mathrm{AffSch} \simeq \mathrm{Ring}^{\mathrm{op}} \quad .$$

(2) *The inclusion* AffSch $\to$ Sch *admits a left adjoint given by sending* $X \in$ Sch *to* $\mathrm{Spec}(\Gamma(X))$. *Concretely a morphism from a scheme* X *to an affine scheme* Y *will factor uniquely through the morphism* $X \to \mathrm{Spec}(\Gamma(X))$.
(3) *A scheme* X *is affine precisely if the morphism* $X \to \mathrm{Spec}(\Gamma(X))$ *is an isomorphism.*
(4) *The scheme* $\mathrm{Spec}(\mathbb{Z})$ *is terminal in the category of schemes, that is for every scheme* X *there is a unique morphism* $X \to \mathrm{Spec}(\mathbb{Z})$.

PROOF. Immediate from the previous statements and Theorem 12.2. □

LEMMA 12.7. *Let* $(X, \mathcal{O}_X)$ *be a scheme and* $U \subseteq X$ *be an open subset. Then* $(U, \mathcal{O}_X|_U)$ *is also a scheme.*

PROOF. To see this note that for a given point $x \in U$ we find an open neighboorhood $V \subseteq X$ such that $(V, \mathcal{O}_X|_V)$ is $\mathrm{Spec}(A)$. But we need to find an affine neighborhood of x that lies in U, i.e in $V \cap U$. The latter is an open set in $\mathrm{Spec}(A)$ so that is contains a principal open which also contains x. This principal open is affine. □

DEFINITION 12.8. *We will denote the affine scheme* $\mathrm{Spec}(k[x_1, , ..., x_n])$ *also by* $\mathbb{A}^n_k$ *for any ground ring* k.

Note that this Definition is slightly dangerous since we have earlier denoted the affine algebraic set k^n by $\mathbb{A}^n_k$. But remember that $\mathrm{Spec}(k[x_1, , ..., x_n])$ is simply the sobrification of the latter and only adds some generic points. Sheaves on the two are the same anyways by Exercise 3 of Sheet 5, so that we can even think of the structure sheaf independently.

EXAMPLE 12.9. We have that for any scheme X

$$\mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^1_{\mathbb{Z}}) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], \mathcal{O}(X)) = \mathcal{O}(X) \ .$$

That is global sections are functions $X \to \mathbb{A}^1_{\mathbb{Z}}$.

EXAMPLE 12.10. Consider the affine scheme $\mathbb{A}^2_k = \mathrm{Spec}(k[X, Y])$ and let U be the open set

$$D(X, Y) = \mathbb{A}^2_k \setminus 0$$

given by the complement of the closed point 0. We consider U as a scheme. By the first exercise of sheet 5 we see that the global sections of the scheme U are given by $k[X, Y]$. Thus if U was affine the morphism

$$U \to \mathrm{Spec}(\Gamma(U)) = \mathbb{A}^2$$

would have to be an isomorphism, which it is not. This morphism is in fact simply the inclusion which is not even surjective.

12.1. Proof of theorem 12.2. Now we would like to proof Theorem 12.2. We have seen that we need to verify that the morphism

$$\mathrm{Hom}\Big((X, \mathcal{O}_X), \mathrm{Spec}(A)\Big) \xrightarrow{\Gamma} \mathrm{Hom}\Big(A, \mathcal{O}_X(X)\Big) \tag{8}$$

is a bijection.
For injectivity assume two maps $f, g : (X, \mathcal{O}_X) \to \mathrm{Spec}(A)$ with corresponding morphisms

$$f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X \qquad g^\sharp : g^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$$

such that f and g induce the same map $\varphi : A \to \mathcal{O}_X(X)$ on global sections. Then for any $x \in X$ we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f^\sharp=\varphi"] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ A_{f(x)} \arrow[r, "f_x^\sharp"] & \mathcal{O}_{X,x} \end{tikzcd}$$

The preimage of the maximal ideal $\mathfrak{m}_{\mathcal{O}_{X,x}} \subseteq \mathcal{O}_{X,x}$ under the counterclockwise composition is $f(x)$. For this we use that $f_x^\sharp$ is a local morphism! From the clockwise composition we see that this preimage does not depend on f but only on φ. Thus we get for each x that $f(x) = g(x)$, i.e. the underlying maps of topological spaces agree. The diagram also shows that the maps $f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$ and $g^\sharp : g^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$ agree on stalks (there is at most one factorization of a map $A \to B$ over $A_{f(x)}$). Thus the maps agree, since any two maps of sheaves that agree on stalks are equal.

Now to prove surjectivity of (8) we have to construct for a given morphism $\varphi : A \to \mathcal{O}_X(X)$ a morphism $(f, f^\sharp)$ of locally ringed spaces. We consider the composition $\varphi_x : A \to \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ and have to set

$$f(x) := \varphi_x^{-1}(\mathfrak{m}_{X,x})$$

This a prime ideal and this way we define a map $f : X \to \mathrm{Spec}(A)$ of topological spaces. We claim that is is continuous. To see this we have to verify that $f^{-1}(D(a))$ for $a \in A$ is open. But we find that

$$\begin{aligned} f^{-1}(D(a)) &= \{x \in X \mid a \notin \varphi_x^{-1}(\mathfrak{m}_{X,x})\} \\ &= \{x \in X \mid \varphi_x(a) \notin \mathfrak{m}_{X,x}\} \\ &= \{x \in X \mid \varphi(a)(x) \neq 0 \text{ in } \kappa(x)\} \end{aligned}$$

By Exercise 3 on sheet 7 this subset is open and therefore f continuous. Now we want to define $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$ for principal opens $D(a)$. To this end we have to construct an extension

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}_X(X) \arrow[d, "\mathrm{res}"] \\ A[1/a] \arrow[r, dashed] & \mathcal{O}_X(f^{-1}(D(a))) \end{tikzcd}$$

which exists precisely if $\varphi(a)$ is invertible on $\mathcal{O}_X(f^{-1}(D(a))$. But invertibility of sections of a sheaf can be checked on stalks: Simply observe that s is invertible if and only if the multiplication by s map $\mathcal{O}_X \to \mathcal{O}_X$ is an isomorphism of sheaves, which can be checked on stalks.

We therefore get our transformation $f^\sharp$ which clearly is compatible with restriction. The last thing we have to verify is that is induces local maps on stalks, so that we have a map of locally ringed spaces.

For this we note that we have by construction the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f^\sharp=\varphi"] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ A_{f(x)} \arrow[r, "f_x^\sharp"] & \mathcal{O}_{X,x} \end{tikzcd}$$

The preimage of the maximal ideal along the counterclockwise composition is $\varphi_x^{-1}(\mathfrak{m}_{X,x})$ which by definition is $f(x)$ so that the claim follows and this finishes the proof.

13. k-valued points

Let k be a field (or even a commutative ring).

Definition 13.1. *For a scheme X we define the k-valued points $X(k)$ to be the set of homomorphisms* $\operatorname{Spec}(k) \to X$.

Example 13.2. Assume that $X = \operatorname{Spec}(R)$ for some ring R. Then we have that $X(k) = \operatorname{Hom}_{\mathrm{Ring}}(R, k)$. For example if $R = \mathbb{Z}[X_1, ..., X_n]/f_1, ..., f_k$ then we have that

$$X(k) = \{x \in k^n \mid f_1(x) = ... = f_k(x) = 0\} .$$

That is $X(k)$ are the solutions to the equations given by the polynomials f_i. This is completely in line with our earlier use of the notation when we considered affine varieties (then we would evaluate at algebraically closed fields k). We could for example evaluate at $k = \mathbb{C}$ or $\mathbb{R}$ and get some intuition what our scheme 'looks like' by evaluating at those points.

Example 13.3. Assume that $X = \operatorname{Spec}(\mathbb{Z}[X]/X^2)$. Then we get that

$$X(k) = \{x \in k \mid x^2 = 0\} = \{0\}$$

for all fields k. Thus by k-valued points we can not even distinguish the schemes $\operatorname{Spec}(\mathbb{Z}[X]/X^2)$ and $\operatorname{Spec}(\mathbb{Z})$.

Definition 13.4. *A scheme X is called* reduced *if for each $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is a reduced ring (i.e. has no nilpotent elements).*

Proposition 13.5. *An affine scheme $\operatorname{Spec}(A)$ is reduced iff A is reduced. A general scheme X is reduced precisely if it admits an open cover $(U_i)_{i \in I}$ by affine reduced schemes U_i. A scheme is reduced precisely if the local rings $\mathcal{O}_{X,x}$ are reduced.*

Proof. If $\operatorname{Spec}(A)$ is reduced then clearly also $A = \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A))$ is reduced. Conversely assume that A is reduced. Then for any $a \in A$ we have that $A[a^{-1}]$ is also reduced: if $(b/a^k)^n = 0$ in $A[a^{-1}]$ then $a^m b^n = 0$ in A for some n and M. This shows that ab is nilpotent in A, so $ab = 0$ and thus $b = 0$ in $A[a^{-1}]$.
For a general open set $U \subseteq \operatorname{Spec}(A)$ we find a cover $U_i = \operatorname{Spec}(A[1/f_i])$ by principal opens and this way we write $\mathcal{O}_{\operatorname{Spec}(A)}(U)$ as a subring of the reduced ring $\prod A[1/f_i]$ which shows that it is also reduced.
For the last part assume $(X, \mathcal{O}_X)$ is reduced. By definition we can cover it by opens $U_i \cong \operatorname{Spec}(A_i)$ and we get that $A_i = \mathcal{O}_X(U_i)$ is reduced.
Conversely assume that X is covered by open, reduced subschemes X_i (e.g. the X_i affine). Then we claim that X is reduced. To see this take $U \subseteq X$ open. Then U is covered by $U \cap X_i$ and all the those sets are open in X_i so that we get $\mathcal{O}_X(U \cap X)$ is reduced. Using the sheaf condition we get that $\mathcal{O}(U)$ is a subring of $\prod_{i \in I} \mathcal{O}(U \cap X_i)$.

For the local statement we observe that a sheaf is reduced precisely if the stalks are reduced. $\square$

EXAMPLE 13.6. For an affine algebraic set V the associated scheme $\mathrm{Spec}(\mathcal{O}(V))$ is reduced.

Recall that for any ring A we can form a reduced ring $A_{red} := A/\sqrt{0}$ which is reduced (as one directly verifies). For a topological space X with a sheaf $\mathcal{O}$ of rings. We define a new sheaf $\mathcal{O}_{red}$ as the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{red}$.

PROPOSITION 13.7. *For a scheme $X = (X, \mathcal{O}_X)$ we have that the ringed space $X_{red} := (X, \mathcal{O}_{X_{red}})$ is also a scheme and we have a canonical morphism $i : X_{red} \to X$ which exhibits X_{red} as the universal reduced scheme with a morphism to X. If $X = \mathrm{Spec}(A)$ then $X_{red} = \mathrm{Spec}(A_{red})$.*

PROOF. For $X = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}\,A})$ we have that as topological spaces $\mathrm{Spec}(A) = \mathrm{Spec}(A_{red})$ since ideals in A_{red} are ideals of A that contain $\sqrt{0}$ and so prime ideals (describing points) and radically closed ideals (describing closed subsets) are the same in A and A_{red}. Under this homeomorphism the open sets $D(f)$ correspond to each other for $f \in A$ (for $f \in \sqrt{0}$ the open set $D(f) = \mathrm{Spec}(A)$ so that we see that there is no difference).
The sheaf $(\mathcal{O}_{\mathrm{Spec}\,A})_{red}$ is given by the sheafifcation of

$$D(f) \mapsto (A[1/f])_{red} = (A_{red})[1/f]$$

thus it agrees with the structure sheaf of $\mathrm{Spec}(A_{red})$.
Now for a general X we observe that for any open $U \subseteq X$ we have that the reduction of $(U, \mathcal{O}_X|_U)$ is given by first reducing X_{red} and then the restriction to U (this follows since sheafifcation commutes with restriction to opens). Therefore we conclude that if U is affine in X then U is also affine in X_{red} which shows that X_{red} is a scheme. Moreover the map of ringed spaces $i : X_{red} \to X$ given by the identity on spaces and $i^\sharp : \mathcal{O}_X \to \mathcal{O}_{X_{red}}$ is a map of locally ringed spaces since it is locally induced by $A \to A_{red}$.
Finally we note that a morphism $Y \to X$ for Y reduced is given by a pair $(f, f^\sharp)$ with

$$f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y.$$

We claim first that $f_*\mathcal{O}_Y$ is reduced (by definition) and therefore this morphism uniquely factors over $\mathcal{O}_{X_{red}}$ which implies that it factors uniquely to a map of ringed spaces $Y \to X^{red}$. We claim that this is a map of locally ringed spaces as one easily verifies (the stalks are the reduction of the stalks). $\square$

In particular this means that we cannot distinguish the schemes X and X_{red} as long as we only consider k valued points for fields k (or more generally reduced rings) since the induced map

$$X_{red}(k) \to X(k)$$

is an isomorphism.
For example we have that $\mathrm{Spec}(\mathbb{Z}[x]/x^2)_{red} = \mathrm{Spec}(\mathbb{Z})$, and thus k-valued points of $\mathrm{Spec}(\mathbb{Z}[x]/x^2)$ and $\mathrm{Spec}(\mathbb{Z})$ agree.
For any commutative ring A we consider the A-valued points $X(A)$ and these assemble together into a functor

$$X(-) : \mathrm{Ring} \to \mathrm{Set}$$

called the *functor of points* of X. We will now see that the functor of points completely determines X:

PROPOSITION 13.8. *The assignment $X \mapsto X(-)$ defines a fully faithful functor*

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) \ .$$

PROOF. We first note that if we replace Ring by Sch then we have for every scheme X the functor $S \in \mathrm{Sch} \mapsto \mathrm{Hom}_{\mathrm{Sch}}(S, X) = \underline{X}$. The Yoneda Lemma implies that this functor

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$$

is fully faithful and our functor is the composition

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\mathrm{res}} \mathrm{Fun}(\mathrm{AffSch}^{\mathrm{op}}, \mathrm{Set})$$

(where we have implicitly identified $\mathrm{AffSch}^{\mathrm{op}} \simeq \mathrm{Ring}$). So our proof roughly proceeds like the proof of the Yoneda lemma: we want to show that every morphism $\eta : \underline{X}|_{\mathrm{AffSch}} \to \underline{Y}|_{\mathrm{AffSch}}$ is induced by a unique morphism $X \to Y$ of schemes. If X is affine this follows from Yoneda's Lemma (applied to AffSch). In general given a natural transformation η

$$\eta_S : \{S \to X\} \to \{S \to Y\}$$

for S affine we note that we can cover X by affines X_i and take the value of the inclusion $\eta_{X_i}(X_i \to X)$ which is a morphism $f_i : X_i \to Y$. These values agree on double intersections $X_i \cap X_j$ and thus using Exercise 2 of sheet 8 we can 'glue' those to a map $f : X \to Y$. We also see that the transformation η is now uniquely determined by this map: for any $j : S \to X$ we can cover S by affine opens S_j such that any S_j maps to some X_i (simply by refining the preimages of X_i by affine opens) and then the composition $\eta(S_j \to S \to X)$ is given by the composite $S_j \to X_i \xrightarrow{f} Y$. Again using the glueing of such maps we see that this already unqiuely determines $S \to Y$. □

Now finally we also want to talk about schemes over some fixed base. To this end we need a general definition from category theory:

DEFINITION 13.9. *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. Then we define the* slice over X *or the* overcategory *to be the category $\mathcal{C}_{/X}$ whose objects are morphisms $Y \to X$ and whose morphisms $Y \to X$ to $Y' \to X$ are commutative diagrams 1*

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

Composition is defined in the evident way. Similarly we define the undercategory *or* slice under X *to be the category $\mathcal{C}_{X/}$ whose objects are morphisms $X \to Z$ and whose morphisms are similar commutative diagrams.*

EXAMPLE 13.10. Let $\mathcal{C} = \mathrm{Ring}$ and k be a commutative ring. Then $\mathrm{Ring}_{k/}$ is equivalent to the category of k-algebras.

EXAMPLE 13.11. If X is terminal then $\mathcal{C}_{/X} \simeq \mathcal{C}$. In particular $\mathrm{Sch}_{/\operatorname{Spec}\mathbb{Z}} \simeq \mathrm{Sch}$.

Definition 13.12. *Let S be a scheme. Then the category $\mathrm{Sch}_{/S}$ is the category of schemes over S. If k is a commutative ring then the category of schemes over k is the category of schemes over $\mathrm{Spec}(k)$, which we also write as Sch_k.*

Note that for a scheme X the structure of a scheme over $\mathrm{Spec}(k)$ is equivalent to a morphism

$$k \to \mathcal{O}_X(X) \ .$$

In particular for any k-algebra A the scheme $\mathrm{Spec}(A)$ is a scheme over k. Moreover using the restrictions $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ we see that a morphism $k \to \mathcal{O}_X(X)$ makes the sheaf $\mathcal{O}_X$ into a sheaf of k-algebras. In fact giving a morphism $X \to \mathrm{Spec}(k)$ is equivalent to lifting the structure sheaf $\mathcal{O}_X$ to a sheaf of k-algebras. Moreofer we get that the category of affine schemes over k (as a full subcategory of Sch_k) is equivalent to the category Alg_k of k-algebras. Thus as a conclusion we could rephrase the definition of a scheme over k as follows:

> A scheme over k is a topological space X, with a sheaf $\mathcal{O}_X$ of k-algebras such that the stalks are local k-algebra and such that the pair $(X, \mathcal{O}_X)$ is locally isomorphism to $\mathrm{Spec}(A)$ for A a k-algebra.

We will often implicitly adopt this point of view.

Proposition 13.13. *The category of AffVar_k for k algebraically closed embedds fully faithfully into the category $\mathrm{Sch}_{/\,\mathrm{Spec}(k)}$ by the functor*

$$V \mapsto \mathrm{Spec}(\mathcal{O}(V)) \ .$$

Proof. The second follows since it is the composition

$$\mathrm{AffVar}_k \to \mathrm{Alg}_k \to \mathrm{Sch}_k$$

of fully faithful functors. □

For a scheme X over $\mathrm{Spec}(k)$ we have that for any k-algebra A we can define the A-valued points $X(A)$ to be the morphisms $\mathrm{Spec}(A) \to X$ over $\mathrm{Spec}(k)$ (i.e. in the category $\mathrm{Sch}_k = \mathrm{Sch}_{/\,\mathrm{Spec}(k)}$. Similarly to the absolute case we obtain a functor

$$\mathrm{Sch}_k \to \mathrm{Fun}(\mathrm{Alg}_k, \mathrm{Set})$$

which is fully faithful.

Warning 13.14. *If we consider a scheme over $\mathrm{Spec}(k)$ as a plain scheme (i.e. forgetting the morphism to $\mathrm{Spec}(k)$) then the two notions of $X(A)$ are different (once relative to k and once not). Therefore it is very important to always keep the basis in mind.*

14. Tensor products and fibre products

The question that we want to discuss in this section is if the category Sch of schemes has (fibre) products. This question is closely related to the question whether the category Alg_k has coproducts. We will see that such a coproduct is given by $(A, B) \mapsto A \otimes_k B$ (yet to be defined). We first want to define R-modules, which are 'vector spaces' over rings instead of fields.

Definition 14.1. *Let R be a ring. An R-module is an abelian group M together with a map*

$$R \times M \to M \qquad (r, m) \mapsto r \cdot m$$

which is associative, distributive and unital, i.e.

$$(rs)m = r(sm) \qquad (r+s)m = rm + sm \qquad r(m+n) = rm + rn \qquad 1m = m$$

for $m, n \in M$ and $r, s \in R$. An R-linear map or R-module map between R-modules M and N is a group homomorphism $f : M \to N$ such that $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$. The category of R-modules is denoted Mod_R.

Example 14.2. If R is a field then R-modules are the same as R-vector spaces. Generally many constructions from vector spaces carry over to R-modules: direct sums $\bigoplus$, products $\prod$, kernels, cokernels, submodules,....
The main difference is that not every R-module admits a basis, i.e. is not necessarily isomorphic to a direct sum $\bigoplus_{i \in I} R$. Therefore we can also not write general R-linear maps as matrices.

Example 14.3. If $R = \mathbb{Z}$ then every abelian group M admits a unique structure of a $\mathbb{Z}$-module: the element $1 \in \mathbb{Z}$ has to act by $1 \cdot m = m$ and therefore $n \geq 0$ by $n \cdot m = m + ... + m$, and $(-n) \cdot m = -(nm)$.
Now we see that the module $\mathbb{Z}/n$ does not admit a 'basis' since every element is n-torsion, so that it is linearly dependent.

Example 14.4. A module over a polynomial ring $R[x]$ is the same as an R-module M together with an R-linear map $f : M \to M$. In particular if $R = k$ is a field then this is a way of expressing a lot of linear algebra (specifically normal forms for linear maps) in the language of modules.

Example 14.5. Let A be a k-algebra for some commutative ground ring k. Then we can consider A as a k-module by the restricted multiplication.

We will restrict attention here to commutative rings R. The notion of a module of course also makes sense for non-commutative rings. In this case one has to distinguish between left and right R-modules though and since we do not need this case we avoid it here.

Definition 14.6. *Let M, N, P be R-modules. An R-bilinear map is a map $\beta : M \times N \to P$ such that $\beta(m, -) : N \to P$ and $\beta(-, n) : M \to P$ are R-linear maps for each fixed m and n.*
A tensor product of M and N is an R-module $M \otimes_R N$ together with an R-bilinear map $i : M \times N \to M \otimes_R N$ such that for any other R-module P the induced map

$$\mathrm{Hom}_R(M \otimes_R N; P) \xrightarrow{i^*} \mathrm{Bil}_R(M \times N; P)$$

is a bijection. In other words: each bilinear morphism $\beta : M \times N \to P$ factors uniquely as

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "i"] & P \\ M \otimes_R N \arrow[ur, dashed] & \end{tikzcd}$$

where the dashed arrow is R-linear. The image $i(m, n)$ is denoted as $m \otimes n$ and called an 'elementary tensor'.

We see that in order to define an R-linear map $f : M \otimes_R N \to P$ we have to define it on elementary tensors, i.e. define $f(m \otimes n) \in P$ for each $m, n \in M$ and check that this is bilinear in m and n. Note that we could equivalently express the universal property as saying that the functor

$$\mathrm{Bil}_R(M \times N; -) : \mathrm{Mod}_R \to \mathrm{Set}$$

is corepresented by $M \otimes_R N$.

EXAMPLE 14.7. Assume that $R = \mathbb{Z}$ and $M = N = \mathbb{Z}$. Then a bilinear map $\beta : \mathbb{Z} \times \mathbb{Z} \to P$ for any abelian group P (aka $\mathbb{Z}$-module) is determined by the value $\beta(1,1)$ since $\beta(m,n) = mn\beta(1,1)$. If follows that the multiplication morphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the universal bilinear morphism, i.e. $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$.
If $M = \mathbb{Z}/n$ and $N = \mathbb{Z}/k$ then we have that a bilinear morphism $M \times N \to P$ is also determined by $\beta(1,1) \in P$, but we need that $\beta(1,1)$ is at the same time n torsion and k-torsion, since

$$n \cdot \beta(1,1) = \beta(n,1) = \beta(0,1) = 0,$$

and analogously in the other variable. But being n and k torsion at the same time is equivalent to being $\gcd(n,k)$-torsion. We conclude that the multiplication map

$$\mathbb{Z}/n \times \mathbb{Z}/k \to \mathbb{Z}/\gcd(n,k)$$

is the universal bilinear map, or said differently $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/k \cong \mathbb{Z}/\gcd(n,k)$.

For the next statement we need the concept of generators and relations descriptions. In general for a given set G there is a free R-module on G given by the direct sum $F(G) = \bigoplus_G R$. The unit vectors e_g are also denoted by g so that G forms a basis of $F(G)$. This free module has a universal property, namely the map

$$\mathrm{Hom}_{\mathrm{Mod}_R}(F(G), P) \to \mathrm{Hom}_{\mathrm{Set}}(G, P)$$

given by restriction to the basis is a bijection. In other words, the functor $F : \mathrm{Set} \to \mathrm{Mod}_R$ is left adjoint to the forgetful functor $\mathrm{Mod}_R \to \mathrm{Set}$.
Now for any subsets Q of elements in $F(G)$ we can form the R-submodule generated by Q and then the quotient which we denote by $F(G)/Q$. This is what we mean by generators and relations descriptions for R-modules. Now we see that

$$\mathrm{Hom}_{\mathrm{Mod}_R}(F(G)/Q, P) = \{ f : G \to P \mid \overline{f}(q) = 0 \quad \forall q \in Q \}$$

where $\overline{f}$ denotes the extension of f to $\overline{f} : F(G) \to P$. Thus for $q = \sum_{i=1}^n a_i q_i \in R$ we have that $0 = \overline{f}(r) = \sum a_i f(q_i)$.

PROPOSITION 14.8. *For every pair of modules M and N there exists a tensor product $M \otimes_R N$.*

PROOF. We define

$$M \otimes_R N := F(M \times N)/Q$$

where Q consists of the relations which make the map $i : M \times N \to F(M \times N)$ bilinear (as we will make explicit in a second). If we denote the basis elements $M \times N \subseteq F(M \times N)$ by $m \otimes n$, then the relations are:

$$\begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n \qquad & m \otimes (n + n') &= m \otimes n + m \otimes n' \\ (rm) \otimes n &= r \cdot (m \otimes n) \qquad & m \otimes (rn) &= r(m \otimes n) \ . \end{aligned}$$

This then does the job since maps $M \otimes_R N \to P$ are by definition determined on $m \otimes n$ and have to be bilinear in m and n. $\square$

Remark 14.9. We see that by definition elements of $M \otimes_R N$ are given by finite R-linear combinations of elementary tensors, i.e. of the form

$$\sum_{i=1}^{n} a_i(m_i \otimes n_i) \ .$$

Note that there are a number of relations between these elements.

Now we want to establish some properties of the tensor product $\otimes$. To this extend we note that for any pair of modules M, N we can consider the set of R-linear homomorphisms $\mathrm{Hom}_R(M, N)$ itself as an R-module, where we define addition and R-multiplication componentwise:

$$(f+g)(m) := f(m) + g(m) \qquad (rf)(m) := rf(m) = f(rm).$$

This defines a functor $\mathrm{Hom}_R : \mathrm{Mod}_R^{\mathrm{op}} \times \mathrm{Mod}_R \to \mathrm{Mod}_R$. Then we see that a R-bilinear map $\beta : M \times N \to P$ is clearly the same as a R-linear map

$$\tilde{\beta} : M \to \mathrm{Hom}_R(N, P)$$

where $\tilde{\beta}(m) : n \mapsto \beta(m, n)$. In other words: $- \otimes_R N$ is as a functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$ left adjoint to the functor $\mathrm{Hom}_R(N, -)$.

Proposition 14.10. *We have natural isomorphisms*

(1) $M \otimes_R N \cong N \otimes_R M$
(2) $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$
(3) $M \otimes_R (N \oplus N') \cong M \otimes_R N \oplus M \otimes_R N'$
(4) $(\mathrm{colim}_I M_i) \otimes_R N \cong \mathrm{colim}_I (M_i \otimes_R N)$
(5) $M \otimes_R (\mathrm{colim}_J N_j) \cong \mathrm{colim}_J (M \otimes_R N_j)$
(6) $(M \otimes_R N) \otimes_R O \cong M \otimes_R (N \otimes_R O)$

Proof. The first isomorphism sends $m \otimes n$ to $n \otimes m$ and is immediately implied by the universal property: bilinear maps $M \times N \to P$ at the same as bilinear maps $N \times M \to P$ by flipping the coordinates.
The second and third statement follow from the fourth and fifth (since the direct sum is the coproduct). By symmetry it is enough to show one of them, lets say (4). This simply says that $- \otimes_R N$ preserves colimits, but that is automatic since it is a left adjoint functor. The last isomorphism simply follows by the observation that bilinear maps $(M \otimes_R N) \times O \to P$ are the same as 'trilinear maps' $M \times N \times O$ and the same for the other way of bracketing (we have implicitly used that the cartesian product is associative). $\square$

Corollary 14.11. *We have that*

$$R^n \otimes_R R^m \cong R^{mn} \ .$$

For any pair of bases $(v_i)_{i \in I}$, $(w_j)_{j \in J}$ *a basis is given by* $(v_i \otimes w_j)_{i \in I, j \in J}$.

Proof. We have that bilinear maps $R^n \times R^m \to P$ are determined by the value on the respective basis element (the same way this works for vector spaces). This way we verify the universal property. $\square$

So if we work with classical vector spaces this determines the tensor product. But it is instructive to think about what happens for linear maps $A : R^n \to R^n$ (represented by a matrix A) and $B : R^m \to R^m$ for the induced map $A \otimes B : R^n \otimes_R R^m \to R^n \otimes_R R^m$. This is sometimes called the Kronecker product.

Now assume that A and B are k-algebras (as usual commutative). Note that the multiplication

$$A \times A \to A \qquad (a, b) \mapsto ab$$

is k-bilinear. Therefore we can consider it as a map $A \otimes_k A \to A$ and similar for B.

PROPOSITION 14.12. *The k-module $A \otimes_k B$ admits the structure of a commutative k-algebra by the multiplication*

$$(a \otimes b) \cdot (a' \otimes b') \mapsto (aa' \otimes bb') \ .$$

The maps

$$\begin{aligned} A &\to A \otimes_k B \qquad a \mapsto a \otimes 1 \\ B &\to A \otimes_k B \qquad b \mapsto 1 \otimes b \end{aligned}$$

exhibit $A \otimes_k B$ as the coproduct in the category of k-algebras.

PROOF. First of all the tensor product is an abelian group. The map

$$(A \otimes_k B) \times (A \otimes_k B) \to (A \otimes_k B) \qquad (a \otimes b) \cdot (a' \otimes b') \mapsto (aa' \otimes bb') \ .$$

is clearly well-defined, in fact it is induced by the quadrilinear map $A \times B \times A \times B$ sending (a, b, a', b') to $aa' \otimes bb'$. Therefore the designed multipliciation is distributive. To see that is is really an algebra we simply need to check that it is also associative, which is obvious.
Now we observe that the maps $A \to A \otimes_k B$ and $B \to A \otimes_k B$ are clearly maps of k-algebras. Moreover in the k-algebra $A \otimes_k B$ every element is a sum of elementary tensors $a \otimes b$ which are in turn the product $(a \otimes 1) \cdot (1 \otimes b)$. Thus every algebra map $A \otimes_k B \to C$ is uniquely determined by the induced maps $A \to C$ and $B \to C$.
Conversely for a pair of map $f_A : A \to C$ and $f_B : B \to C$ we get an induced map

$$f : A \otimes_k B \to C \qquad (a \otimes b) \mapsto f_A(a) \cdot f_B(b)$$

since the right hand term is bilinear. Now this is a map of algebras since

$$\begin{aligned} f((a \otimes b) \cdot (a' \otimes b')) &= f(aa' \otimes bb') \\ &= f_A(aa') \cdot f_B(bb') \\ &= f_A(a) f_A(a') f_B(b) f_B(b') \\ &= f(a \otimes b) \cdot f(a' \otimes b') \ . \end{aligned}$$

Note that in the last step we have crucially used the commutativity of C. □

EXAMPLE 14.13. We have that $R[x] \otimes_R R[y] \cong R[x, y]$. This simply follows from the universal properties since $R[x] \otimes_R R[y]$ is the coproduct and clearly $R[x, y]$ is the coproduct by the universal property. But one can also clearly see that since $R[x]$ has a basis consisting of powers of x and $R[y]$ has a basis consisting of powers of y we get that the tensor product $R[x] \otimes_R R[y]$ has a basis consisting of elements of the form $x^i \otimes y^j$. These are mapped to the monomial $x^i y^j$.

PROPOSITION 14.14. *For a diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \\ B & \end{tikzcd}$$

of commutative rings we have that $A \otimes_C B$ is the pushout, i.e. the colimit of this diagram in the category Ring *(here we forget that it is actually a C-algebra).*

PROOF. First note that for any commutative ring D a map of rings $f : A \otimes_C B \to D$ makes D into a C-algebra through the composition $C \to A \otimes_C B \to D$. With this C-algebra structure on D the map F becomes a map of C-algebras. This shows that a map of rings f is the same as a pair consisting of a map $C \to D$ and a C-linear map $f : A \otimes_C B \to D$. The latter is by the previous result the same as a pair of C-linear maps $f_A : A \to D$ and $f_B : B \to D$, in other words maps of rings such that the composites $C \to A \to D$ and $C \to B \to D$ agree with the map $C \to D$ and in particular agree with each other. All the data together is simply given by two maps $A \to D$ and $B \to D$ such that the two composites $C \to D$ agree. This is the universal property of the pushout. $\square$

Dually a limit of a diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

is called a pullback. In fact, such a limit Z can also be depicted by a square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

which we then call a pullback square. We often also write $Z = X \times_S Y$.
If I is a finite category (that is finitely many objects and finitely many morphisms) then we say that a functor $I \to \mathcal{C}$ is called a finite diagram and its limit is called a finite limit. We say that $\mathcal{C}$ admits all finite limits if all finite diagrams admit a limit.

THEOREM 14.15. *The category of schemes has all finite limits. For a diagram*

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(C) \end{tikzcd}$$

the fibre product is $\mathrm{Spec}(A \otimes_C B)$.

Proving this statement will take some time (and additional Lemmas). But let us first make a remark.

REMARK 14.16. In general arbitrary limits of schemes do not exist (I initially made unfortunately a wrong claim to that extend!). Note that by the universal property the functor of points of the pullback can be described as the pullback

$$(X \times_S Y)(R) = X(R) \times_{S(R)} Y(R) .$$

This is a very convenient description that we will employ. The main question really is, if this pullback of functors is represented by a scheme.

We will first need a lemma for this statement.

LEMMA 14.17. *Assume that a category $\mathcal{C}$ has a terminal object and all pullbacks. Then it has all finite limits.*

PROOF. If $\mathcal{C}$ has pullbacks and a terminal object pt, then it also has finite products, since $A \times B$ is the pullback $A \times_{\mathrm{pt}} B$. Moreover we claim that $\mathcal{C}$ also has equalizers, namely the equalizer of $f, g : A \to B$ can be written as the pullback

$$\begin{tikzcd} & B \arrow[d, "\Delta"] \\ A \arrow[r, "(f,g)"] & B \times B \end{tikzcd}$$

Every finite limit $\lim_I X_i$ can be written as an equalizer of two maps $\prod_{i \in I} X_i \to \prod_{i \to j} X_i$ $\square$

So in order to prove the theorem we need to verify that Sch has a terminal object and pullbacks. We have already seen that Sch has a terminal object, namely $\mathrm{Spec}(\mathbb{Z})$, so that it is enough to show that all pullbacks exist. In the previous statement we have seen that the square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & B \otimes_C A \end{tikzcd}$$

is a pushout in Ring. Thus the square

$$\begin{tikzcd} \mathrm{Spec}(B \otimes_C A) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(C) \end{tikzcd}$$

is a pullback in AffSch. But the inclusion AffSch $\to$ Sch is right adjoint with left adjoint given by $\mathrm{Spec}(\Gamma)$ (Proposition 12.6). Thus it preserves limits and we can conclude that this square is also a pullback in Sch.
For a general diagram of schemes

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "q"] & S \end{tikzcd}$$

we can find an indexing set I and open affine covers $(X_i)_{i \in I}$ of X,$(Y_i)_{i \in I}$ of Y and $(S_i)_{i \in I}$ of S such that $p(X_i) \subseteq S_i$ and $q(Z_i) \subseteq S_i$ for all i. To do this simply start with an open affine cover $(S_i)_{i \in I}$ of S and let the covers of X and Y be refinements of the cover of S. They would a priori be defined for more complicated indexing sets than I but we can without loss of generality make the cover of S very redundant and insert many copies for the same index.
We get induced diagrams

$$\begin{tikzcd} & X_i \arrow[d, "p"] \\ Y_i \arrow[r, "q"] & S_i \end{tikzcd}$$

and we can form the pullbacks $Z_i := X_i \times_{S_i} Y_i$ which are affine schemes. The idea now is to form a scheme Z which has an open cover given by the Z_i's. To carry this out we need to discuss the 'glueing' of schemes.

This leads us to discuss colimits of schemes. Recall that for a family of topological spaces X_i we can form a new topological space

$$\coprod X_i$$

which is as a set given by the disjoint union of the X_i. The topology is determined by declaring a subset $U \subseteq \coprod X_i$ to be open, precisely if all the intersections $U \cap X_i$ are open in X_i. Said differently: an open set is a disjoint union of open subsets $U_i \subseteq X_i$. It is straighforward to check that this is a coproduct in the category of topological spaces and continuous maps.

LEMMA 14.18. *Let $X_i = (X_i, \mathcal{O}_{X_i})$ be a family of schemes. Then the coproduct in* Sch *exists and is given by the disjoint union of topological spaces*

$$X = \coprod X_i$$

with the structure sheaf $\mathcal{O}_X$ given by $\mathcal{O}(\coprod_i U_i) = \prod_i \mathcal{O}_{X_i}(U_i)$.

PROOF. This clearly describes a sheaf, for example since it can be described as $\prod_I (i_i)_*(\mathcal{O}_{X_i})$ where $i_i : X_i \to X$ is the inclusion. Clearly $(X, \mathcal{O}_X)$ has a cover by affines, given by covering each of the X_i's by affines (as the restriction of $\mathcal{O}$ to $X_i \subseteq \coprod X_i$ is given by $\mathcal{O}_{X_i}$). The universal property can be checked directly: a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f = (f_i) : X \to Y$ with $f_i : X_i \to Y$ together with a morphism of sheaves

$$f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X) = f_* \prod_I (i_i)_*(\mathcal{O}_{X_i}) = \prod (f_i)_*(\mathcal{O}_{X_i})$$

which is a family of morphisms $f_i^\sharp$ as desired. Thus so far we have shown that $(X, \mathcal{O}_X)$ is the coproduct in locally ringed spaces. To see that it is indeed the corproduct in locally ringed spaces we note that the condition on stalks is a local condition, so for f translates into the corresponding condition for each f_i. $\square$

LEMMA 14.19. *Assume that we have a diagram of schemes*

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle j}\downarrow & & \\ Y & & \end{array}$$

of schemes where both maps are open immersions. Then the pushout $X \amalg_U Y$ in the category of schemes exists and the maps $X \to X \amalg_U Y \leftarrow Y$ are open immersions.

PROOF. We form the pushout in topological spaces as follows: the space is given by the disjoint union $X \amalg Y$ modulo the equivalence relation that $i(x) \sim j(x)$ for $x \in U$. One should think of this as the 'union' of X and Y. Now we equip this space with a topology where a subset in $X \amalg_U Y$ is open precisely if its intersection with X and Y is open. It is not hard to check that this is a topological space and that it is in fact the pushout of this diagram in the category of topological spaces.
Now we would like to form the pushout in schemes. To do this we denote the inclusions of topological spaces $i : X \to X \amalg_U Y$, $j : Y \to X \amalg_U Y$ and $k : U \to X \amalg_U Y$. Then we can form the sheaves $i_*(\mathcal{O}_X)$, $j_*(\mathcal{O}_Y)$ and $k_*(\mathcal{O}_U)$ and we get

induced maps

$$\begin{tikzcd} & i_*(\mathcal{O}_X) \arrow[d] \\ j_*(\mathcal{O}_Y) \arrow[r] & k_*(\mathcal{O}_U) \end{tikzcd} .$$

We now form then pullback of sheaves to obtain a sheaf $\mathcal{O}_Z$. it is not hard to see that $\mathcal{O}_Z$ is a sheaf that restricts to $\mathcal{O}_X$ and $\mathcal{O}_Y$ on the respective open subsets of $Z = X \cup_U Y$. Thus Z is in fact a scheme. Moreover by construction we find that it is the pushout in schemes: a map of ringed spaces

$$(f, f\sharp) : (Z, \mathcal{O}_Z) \to (W, \mathcal{O}_W)$$

is given by maps $f_X : X \to Y$ and $f_y : X \to Y$ together with a map of sheaves

$$\mathcal{O}_W \to f_*(\mathcal{O}_Z) \ .$$

Now using the pullback description of $\mathcal{O}_Z$ (and the fact that f_* preserves pullbacks) we see that this can indeed be described as a pair of maps $\mathcal{O}_W \to (f_X)_*\mathcal{O}_Z$ and $\mathcal{O}_W \to (f_Y)_*\mathcal{O}_Z$ so that the two resulting maps agree. This shows that we have the pushout in ringed spaces. But again the locality condition can be checked locally in Z so that it is also the pushout in locally ringed spaces. $\square$

Remark 14.20. For any I-indexed diagram $(X_i, \mathcal{O}_i)$ of ringed spaces we can form the colimit in ringed spaces as follows:

$$(\operatorname{colim}_I X_i, \lim_I (j_i)_*\mathcal{O}_i)$$

where $j_i : X_i \to \operatorname{colim} X_i$ is the canonical map. This follows exactly as in the previous statements. If the X_i form an open cover of X then it follows that it is a locally ringed space and in fact the colimit in locally ringed spaces. This is what we have used so far.

Note that for every scheme X we have by definition an open cover by affines $X_i \subseteq X$. One can wonder how to reconstruct X from this datum:

Proposition 14.21. *For every scheme X and every open cover X_i we have that X is the coequalizer of the diagram*

$$\textstyle\coprod_{i,j}(X_i \cap X_j) \rightrightarrows \coprod X_i$$

where the first map is induced by the inclusion $X_i \cap X_j \to X_i$ and the second by the inclusion $X_i \cap X_j \to X_j$.

Proof. This follows from the sheaf condition for maps of schemes proven in Exercise 2 of sheet 8. $\square$

This tells us that we can recover X from the X_i's if we additionally remember the double intersections $X_i \cap X_j$ and how this sits inside of X_i and X_j. Let us denote the image of $X_i \cap X_j$ in X_i by $U_{ij} \subseteq X_i$ and the image in X_j by U_{ji}. Then we have an isomorphism

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

which is really the identity in our identifications (but categorically this does not make sense). Moreover if we consider the triple intersection

$$X_i \cap X_j \cap X_k = U_{ij} \cap U_{jk} = U_{ji} \cap U_{jk} = U_{ki} \cap U_{kj}$$

as a subset of X_i then we can compare the maps $\varphi_{jk}\varphi_{ij}$ and φ_{ik} which have target X_k or rather an open subset of X_k. These maps agree.

THEOREM 14.22 (Glueing schemes). *Assume that we have a family of schemes X_i for $i \in I$, for each pair i, j an open subscheme $U_{ij} \subseteq X_i$, and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$. Assume that $U_{ii} = X_i$, that $\varphi_{ii} = \mathrm{id}$ and that the following* cocycle condition *is satisfied: for each triple i, j, k we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ and the diagram*

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

commutes. Then there is a scheme X which has X_i up to isomorphisms an open cover and with intersections $U_{ij} = X_i \cap X_j$.

PROOF. Using similar arguments to what we have done before (See Remark 14.20) it is enough to construct the underlying toplogical space of X and observe that X_i is an open cover. Then we can form the structure sheaf by taking the limit of the pushforwards of the sheaves from U_{ij}. Similar to the case of pushouts but a bit more technically involved. But the topological space can be formed as the quotient of $\sqcup X_i$ by an equivalence relation (which identified $x \in U_{ij}$ with $\varphi_{ij}(x)$). Then one has to check that the maps $X_i \to X$ are open which is an exercise in quotient topologies that we omit here. The rest follows as before. $\square$

PROOF OF THEOREM 14.15. Assume that the pullback $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme. Then we claim that the pullback $U \times_S Y$ exists and is given by the open subscheme of $X \times_S Y$ induced by $p^{-1}(U)$ where $p : X \times_S Y \to X$ is the projection. This follows easily by verification of universal properties: unwinding the definitions we see that a morphisms $Z \to p^{-1}U$ is given by a pair of morphisms as desired.

Now assume that for an open covering X_i of X the fibre products $X_i \times_S Y$ exists. Then the fibre product $X \times_S Y$ exists as we will argue now. Indeed we can simply build it using the cocycle description: we define the double intersections $U_{ij} := X_i \cap X_j \subseteq X_i$ as before and then we have that

$$U_{ij} \times_S Y$$

is open in $X_i \times_S Y$ (by the first part of the proof) and moreover we have isomorphisms $\varphi_{ij} : U_{ij} \times_S Y \cong U_{ji} \times_S Y$ satisfying the cocycle conditions induced by the respective isos $U_{ij} \to U_{ji}$.

Thus we can use Theorem 14.22 to 'glue' those $X_i \times_S Y$ together to a scheme $X \times_S Y$ which has the $X_i \times_S Y$ as a cover by open subschemes. We claim that $X \times_S Y$ is indeed the fibre product. To this end we have to verify the universal property: for a map

$$f : K \to X \times_S Y$$

from a scheme K one can find an open cover K_i such that each K_i maps to one of the opens $X_i \times_S Y$. Namely we simply take the preimages of the opens. But then using that maps of schemes are a sheaf the map f is unqiuely determined by the maps $f_i : K_i \to X_i \times_S Y$. Now by the universal property of the pullbacks $X_i \times_S Y$ we see that each f_i is determined by maps $g_i : K_i \to X_i$ and maps $h_i : K \to Y$ so that the resulting maps $K_i \to S$ agree. Now using that these maps have to agree on double intersections $K_i \cap K_j$ we see that we obtain maps

$$g : K \to X \qquad \text{and} \qquad h : K \to Y$$

so that the maps $K \to S$ agree. Vice versa these maps uniquely determine the map f which shows the claim.

After this discussion we see that we can reduce the claim in the X-variable to an affine scheme X_i. A similar argument reduces it in the Y and S variable to affine schemes. But the fact that the pullback of affine maps exist was argued before (using that it is given by the spectrum of the tensor product). This then finishes the proof. □

15. Projective space

In this section we will introduce the scheme $\mathbb{P}^n$ called n-dimensional projective space. Its $\mathbb{R}$-valued points are given by classical real and complex projective space $\mathbb{P}^n(\mathbb{R})$ or $\mathbb{P}^n(\mathbb{C})$ (usually denoted $\mathbb{RP}^n$ and $\mathbb{CP}^n$). Let first recall the definition of those two:

Definition 15.1. *We let k be a field (here $\mathbb{R}$ or $\mathbb{C}$). Then we set*

$$\mathbb{P}^n(k) := \frac{k^{n+1} \setminus \{0\}}{\sim}$$

where the equivalence relation is given by

$$(x_0, ..., x_n) \sim \lambda(x_0, ..., x_n) \qquad \text{for } \lambda \in k^\times$$

It is the orbit relation for the $k^\times$ action on $k^{n+1} \setminus \{0\}$ given by multiplication. Thus we also have

$$\mathbb{P}^n(k) = \frac{k^{n+1} \setminus \{0\}}{k^\times} = \{\text{One dimensional subspaces of } k^{n+1}\}$$

We write an element as $(x_0 : \ldots : x_n)$ call this representation homogenous coordinates.

Example 15.2. Lets try to understand $\mathbb{P}^1(\mathbb{R}) = (\mathbb{R}^2 \setminus 0)/\mathbb{R}^\times$. We have the subset

$$U_0 = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{R}) \mid x_0 \neq 0\}$$

The map $\mathbb{R} \to U_1$ given by $x \mapsto (1 : x)$ is a bijection with inverse $(x_0 : x_1) \mapsto x_1/x_0$. Then we have that

$$\mathbb{P}^1(\mathbb{R}) = U_1 \cup [0 : 1] = \mathbb{R}^1 \cup \{\infty\}$$

so that we can think of $\mathbb{P}^1(R)$ as adding to $\mathbb{R}$ one further point $[0 : 1]$ which one might call ∞. If we equip $\mathbb{P}^1(R)$ with the quotient topology it is homeomorphic to S^1. There is another open open set $U_0 = \{(x_0 : x_1) \mid x_1 \neq 0\}$ isomorphic to $\mathbb{R}$ and they cover $\mathbb{P}^1(\mathbb{R})$.

Example 15.3. We similary have that $\mathbb{P}^2(R)$ has an open set

$$U_0 = \{(x_0 : x_1 : x_2) \mid x_0 \neq 0\} \cong \mathbb{R}^2$$

and we have that

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{P}^1$$

so that we can think of $\mathbb{P}^2$ by enlargening $\mathbb{R}^2$ and further adding one point at ∞ for each direction. Now we can find three open sets U_0, U_1, U_2 isomorphic to $\mathbb{R}^2$.

EXAMPLE 15.4. In general we have that

$$\mathbb{P}^n(k) = k^n \cup \mathbb{P}^{n-1}(k) = \ldots = k^n \cup k^{n-1} \cup k^{n-2} \cup ... \cup k^0$$

We can cover $\mathbb{P}^n(k)$ by 'charts'

$$U_i = \{(x_0 : ... : x_n) \mid x_i \neq 0\} \cong k^n$$

where $(x_0 : ... : x_n) \mapsto (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$. We define the functions

$$X_{s,i} : U_i \to k^n \xrightarrow{pr} k$$

for $s \neq i$, which is given by $(x_0 : ... : x_n) \mapsto x_s/x_i$. Accordingly, we set $X_{i,i} = 1$. On double intersections $U_i \cap U_j$ we have the equality

$$X_{s,i} = x_s/x_i = x_s/x_j \cdot (x_i/x_j)^{-1} = X_{s,j} X_{i,j}^{-1}$$

for all i, j, s. This description on fact works for all fields k.

Our goal is to construct a scheme $\mathbb{P}^n$ such that for every field the k-valued points are given by $\mathbb{P}^n(k)$ as defined above. We will do this by gluing together Plücker-coordinate sheets (using Theorem 14.22 which allows to glue schemes).

DEFINITION 15.5. *For $i = 0, .., n$ we set*

$$U_i := \operatorname{Spec}\left(\frac{\mathbb{Z}[X_{0,i}, \ldots, X_{n,i}]}{X_{i,i} - 1}\right) \cong \mathbb{A}^n \ .$$

We have open subsets

$$U_{i,j} = D(X_{i,j}) \subseteq U_i$$

given by $\operatorname{Spec}(\mathbb{Z}[X_{0,i}, ..., X_{j,i}^{\pm}, ..., X_{n,i}]/(X_{i,i} - 1))$. *We define isomorphisms*

$$\varphi_{i,j} : U_{j,i} \xrightarrow{\simeq} U_{i,j}$$

which are given by maps of coordinate ring in the opposite direction which is defined as

$$\varphi_{i,j}(X_{s,i}) = X_{s,j} \cdot X_{i,j}^{-1} \ .$$

where we set $X_{j,j} = 1$. We have the cocycle condition

$$\begin{aligned} \varphi_{j,k} \circ \varphi_{i,j}(X_{s,i}) &= \varphi_{j,k}(X_{s,j} \cdot X_{i,j}^{-1}) \\ &= \varphi_{j,k}(X_{s,j}) \cdot \varphi_{j,k}(X_{i,j})^{-1} \\ &= (X_{s,k} X_{j,k}^{-1}) \cdot (X_{i,k} X_{j,k}^{-1})^{-1} \\ &= \varphi_{i,k}(X_{s,i}) \ . \end{aligned}$$

Thus we can glue a scheme from those which we call $\mathbb{P}^n$.

The scheme $\mathbb{P}^n$ has by definition open subsets $U_i \subseteq \mathbb{P}^n$ which are isomorphic to $\mathbb{A}^n$ glued along the open subsets $U_{i,j} = U_{j,i}$. In particular for a field k we have that

$$\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i(k) = \bigcup_{i=0}^n k^n = \mathbb{P}^n(k)$$

by what we have said before. Here we have used that $\operatorname{Spec}(k)$ is simply a point. (I.e. any map $\operatorname{Spec}(k) \to \mathbb{P}^n$ factors through one of these open subspaces.) For a general ring R we do *not* have that

$$\mathbb{P}^n(R) = R^{n+1}/\sim \ .$$

Our goal is to give a description of $\mathbb{P}^n(R)$ for an arbitrary ring R now (we have seen that this describes the whole scheme uniquely but the k-valued points don't).

DEFINITION 15.6. *Let L be an R-module. Then L is called invertible if there exists another R-module L' such that $L \otimes_R L' \cong R$ as R-modules.*

EXAMPLE 15.7. Let $R = k$ be a field. Then every k-module is of the form $\bigoplus_I k$. The tensor product of $\bigoplus_I k$ and $\bigoplus_J k$ is given by $\bigoplus_{I\times J} k$. We see that a k-module L can only be invertible if it is 1-dimensional, thus k is the only invertible k-module (up to isomorphism).

Assume that M is an R-module and that $\varphi : R \to S$ is a map of rings. Then the tensor product $\varphi^*(M) := M \otimes_R S$ is an S-module by

$$S \times (M \otimes_R S) \to M \otimes_R S \qquad s \cdot (m \otimes s') = m \otimes ss' \ .$$

Moreover we have for R-modules M and N:

$$(M \otimes_R S) \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S \ .$$

EXAMPLE 15.8. If L is an invertible R-module and $\varphi : R \to S$ a map of rings then we have that $\varphi^*(L) = L \otimes_R S$ is an invertible S module since we have

$$(L \otimes_R S) \otimes (L' \otimes_R S) \cong (L \otimes_R L') \otimes_R S \cong R \otimes_R S \cong S.$$

Note that if R is a ring and $I \subseteq R$ and ideal, then we can consider I as an R-module, in fact an R-submodule of R. Note that I is precisely isomorphic as an R-module to R if it is a principal ideal generated by a non-zero-divisor (then it is of the form (a) and a is a basis). In particular all principal ideals generated by non-zero-divisors are invertible R-modules, but boring ones as they are isomorphic to R. But one can give non-trivial examples of invertible R-modules along those lines.

EXAMPLE 15.9. Consider the ring $R = \mathbb{Z}[\sqrt{-5}] \subseteq \mathbb{Q}(\sqrt{-5}) = \mathbb{Q} \oplus \mathbb{Q}\sqrt{-5} \subseteq \mathbb{C}$ given by

$$\mathbb{Z} \oplus \mathbb{Z}\sqrt{-5} = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \ .$$

This is a standard example for a ring that is not a UFD as one has:

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3 \ .$$

Then we consider the ideal

$$I = (2, 1 + \sqrt{-5}) = (2, 1 - \sqrt{5}) \subseteq R$$

PROPOSITION 15.10. *The R-module I is invertible as an R-module and not isomorphic to R.*

PROOF. We consider $I \otimes_R I$ and construct a map of R-modules

$$I \otimes_R I \to (2) \subseteq R \qquad (i, j) \mapsto i \cdot j$$

which works since $(1 + \sqrt{-5})^2 = 1 + 2\sqrt{-5} - 5 = 2 \cdot (-2 + \sqrt{-5})$. We claim that this map is an isomorphism. This then shows that I is invertible (with inverse I) since $(2) \cong R$.
To verify that this map is an isomorphism we note that we also have a map of R-modules

$$(2) \to I \otimes_R I \qquad 2 \mapsto (1 + \sqrt{-5}) \otimes (1 - \sqrt{-5}) - 2 \otimes 2 \ .$$

One easily verifies that the composition (2) $\mapsto$ (2) is the identity. Conversely to verify that the composition $I \otimes_R I \to I \otimes_R I$ is the identity it suffices to write $I \otimes_R I$ as $(2, 1+\sqrt{-5}) \otimes (2, 1-\sqrt{-5})$ and check what the map does on the four generators

$$2 \otimes 2 \quad 2 \otimes (1-\sqrt{-5}) \quad (1+\sqrt{5}) \otimes 2 \quad (1+\sqrt{5}) \otimes (1-\sqrt{-5}) \ .$$

We treat the case of the fourth generator, the other ones are similar but easier:

$$(1+\sqrt{5}) \otimes (1-\sqrt{-5}) \mapsto 6 \mapsto 3\Big((1+\sqrt{-5}) \otimes (1-\sqrt{-5}) - (2 \otimes 2)\Big)$$

In order to verify that this generator maps to itself we consider the difference

$$\begin{aligned} 2\Big((1+\sqrt{-5}) \otimes (1-\sqrt{-5})\Big) - 3(2 \otimes 2) &= \Big(2(1+\sqrt{-5}) \otimes (1-\sqrt{-5})\Big) - (2 \otimes 3 \cdot 2) \\ &= \Big(2 \otimes (1+\sqrt{-5})(1-\sqrt{-5})\Big) - (2 \otimes 6) \\ &= (2 \otimes 6) - (2 \otimes 6) = 0 \ . \end{aligned}$$

For the non-trviality we have to show that I is not principal. A generator $a + b\sqrt{-5}$ would divide 2, thus the conjugate $a - b\sqrt{-5}$ would also divide 2. Therefore the integer $(a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2$ would divide 4. The only possibilities for this are $b = 0$ and $a = \pm 2$ or $a = \pm 1$. Neither of those generate the ideal. $\square$

Now we would like to describe the set $\mathbb{P}^n(R)$ for a ring R. Recall that for a field we have

$$\mathbb{P}^n(k) = \{\text{One dimensional subspaces of } k^{n+1}\}$$

The question is what the 'correct' generalization of this description for arbitrary rings R is. The idea is to consider invertible modules L as 1-dimensional (we will be even more precise about this analogy eventually). Another difference between rings and fields is that for an R-module M with a submodule $N \subseteq M$ we do not necessarily have a complement $N' \subseteq M$ such that $N \oplus N' \cong M$. A neccessary and sufficient condition for this is that the inclusion $i : N \to M$ has a retract, i.e. a map $r : M \to N$ such that $ri = \mathrm{id}_N$. For example the submodule $2\mathbb{Z} \subseteq \mathbb{Z}$ does not have a complement since there is no retract from $\mathbb{Z}$ to $2\mathbb{Z}$.

Definition 15.11. *Let M be an R-module. We say that a submodule $N \subseteq M$ is 1-dimensional and complementable if N is invertible as an R-module and it admits a complement as a submodule (equivalently a retract).*

Theorem 15.12. *For every ring R we have a bijection*

$$\mathbb{P}^n(R) \cong \{\text{One dimensional complementable subspaces of } R^{n+1}\} \ .$$

We will prove this theorem after exploring the statement a bit. First we have to see that the right hand side of the bijection is indeed a covariant functor in R. For a 1-dimensional complemented submodule $L \subseteq R^{n+1}$ and a map of rings $\varphi : R \to S$ we consider the induced map $\widetilde{\varphi} : R^{n+1} \to S^{n+1}$ given by levelwise application of φ. Then we can take the image $\widetilde{\varphi}(L) \subseteq S^{n+1}$. It is clearly an additive subgroup and even an R-submodule, but not necessarily an S-submodule. Thus we can form the S-submodule generated by this set

$$\varphi^*(L) = S \cdot \widetilde{\varphi}(L)$$

given by linear combinations of S-multiples of $\widetilde{\varphi}(L)$.

Lemma 15.13. *The S-submodule $\varphi^*(L)$ is 1-dimensional and complemented. In fact, as an S-module it is isomorphic to $L \otimes_R S$*

Proof. Note that in general for an injective map of R-modules $N \to M$ the induced map $N \otimes_R S \to M \otimes_R S$ is a map of S-modules which is not necessarily injective. However, if i admits a retract (i.e. its image is a complements submodule) then the induced map $N \otimes_R S \to M \otimes_R S$ also admits a retract induced from the retract of the initial map by functoriality of tensoring. Thus it is still injective. Now we take the inclusion $i : L \to R^{n+1}$ and observe that the induced map $L \otimes_R S \to R^{n+1} \otimes_R S = S^{n+1}$ is an S-linear map which admits a retract, in particular is injective. Now to prove that $\varphi^*(L) \cong L \otimes_R S$ we simply observe that $\varphi^*(L)$ is (up to isomorphism) a submodule of S^{n+1} by what we have just said and it is clearly contained in $\varphi^*(L)$. Moreover it contains the image $\widetilde{\varphi}(L)$ and therefore the two agree.[5] Now this immediately implies that $\varphi^*(L)$ admits is complemented and also by Example 15.8 that it is invertible. $\square$

Note that this also says that our two uses of the notation $\varphi^*(L)$ agree (here and before Example 15.8).

Remark 15.14. This makes

$$\{\text{One dimensional complementable subspaces of } R^{n+1}\}$$

into a functor Ring $\to$ Set and as such it is naturally isomorphic to the functor of points of $\mathbb{P}^n$. Since the functor of points uniquely determines a scheme one can in fact use this as a definition of $\mathbb{P}^n$. Then one only has to check that this functor of points is representable by a scheme. Grothendieck follows this path in EGA (except that he uses a dual description of complemented subspaces as certain quotients of R^{n+1} by passing to the 'dual map' of the inclusion).

Now note that for any scheme X we have that $X(R) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), X)$ is a sheaf on $\mathrm{Spec}(R)$, in particular if we have an open cover V_i by principal opens $V_i = \mathrm{Spec}(R[f_i^{-1}]) = \mathrm{Spec}(R_i)$ with double intersections

$$V_i \cap V_j = \mathrm{Spec}(R[1/f_i, 1/f_j]) = \mathrm{Spec}(R_{i,j})$$

then the diagram

$$X(R) \longrightarrow \textstyle\prod_i X(R_i) \rightrightarrows \prod_{i,j} X(R_{ij})$$

is an equalizer. So for any functor F : Ring $\to$ Set a necessary condition for being a functor of points is that this condition is satisfied.

Definition 15.15. *We say that a functor F : Ring $\to$ Set satisfies* descent *if for any open cover $V_i = \mathrm{Spec}(R_i)$ of $\mathrm{Spec}(R)$ by principal opens as above with double intersection $\mathrm{Spec}(R_{ij})$ the diagram*

$$X(R) \longrightarrow \textstyle\prod_i X(R_i) \rightrightarrows \prod_{i,j} X(R_{ij})$$

is an equalizer, or equivalently of for any R the restriction of F to $\mathrm{Open}(\mathrm{Spec}(R))$, i.e the induced presheaf in $\mathrm{PSh}_{\mathcal{B}}(\mathrm{Spec}(R); \mathrm{Set})$, is a sheaf.

Proposition 15.16. *Any functor of points of a scheme satisfies descent.*

[5] Said differently the map

$$L \otimes_R S \to \varphi^*(L) \quad (x \otimes s) \mapsto s \cdot \widetilde{\varphi}(l)$$

which is a map of S-modules is an isomorphism since both inject into S^{n+1} and have the same image.

Thus a necessary condition for a natural equivalence

$$\mathbb{P}^n(R) \cong \{\text{One dimensional complementable subspaces of } R^{n+1}\} .$$

as in Theorem 15.12 is that the right hand functor satisfied descent. This is indeed the case:

LEMMA 15.17. *The functor*

$$R \mapsto \{\textit{One dimensional complementable subspaces of } R^{n+1}\}$$

satisfies descent.

PROOF. We will prove this in the next section as a special case of descent for general modules and maps of modules. □

We finally need one more Lemma in order to give the proof of Theorem 15.12 which we will also prove later.

LEMMA 15.18 (Nakayama's Lemma, prime ideal version). *Assume that M is a finitely generated R-module with elements $m_1, ..., m_n \in M$ such that for some prime ideal $x \in \operatorname{Spec}(R)$ the images of the elements $m_1, ..., m_n$ generate $M \otimes_R \kappa(x)$ as a $\kappa(x)$-vector space. Then there is some $f \in R$ with $f \notin x$ such that the elements $m_1, ..., m_n$ already generate $M[1/f]$ as a $R[1/f]$-module.*

PROOF OF THEOREM 15.12. We want to to prove the Theorem by giving explicit maps both ways:
Assume that we have a map $g : \operatorname{Spec}(R) \to \mathbb{P}^n$. We choose an open covering $(V_i)_{i \in I}$ of $\operatorname{Spec}(R)$ together with a map $\pi : I \to \{0, .., n\}$ such that $g(V_i) \subseteq U_{\pi(i)}$ where the sets

$$U_0, ..., U_n \subseteq \mathbb{P}^n$$

are the standard opens of $\mathbb{P}^n$. Thus we have maps $V_i \to U_{\pi(i)} \subseteq \mathbb{P}^n$. We assume that $V_i = \operatorname{Spec}(R_i)$ is standard open. The map is thus described by a map of rings

$$g_i : \frac{\mathbb{Z}[X_{0,\pi(i)}, \ldots, X_{n,\pi(i)}]}{X_{\pi(i),\pi(i)} - 1} \to R_i$$

i.e. a sequence of elements $(g_i(X_{0,\pi_i}), \ldots, g_i(X_{n,\pi(i)}))$ with $g_i(X_{\pi(i),\pi(i)}) = 1$. We then consider the submodule

$$L_i = R \cdot v_i \subseteq R_i^{n+1}$$

spanned by the vector $v_i = \big(g_i(X_{0,\pi(i)}), g_i(X_{1,\pi(i)}), ..., g_i(X_{n,\pi(i)})\big)$ and claim that this is a complementable 1-dimensional subspace. To see that it is complementable we note that it has a complement given by

$$L_i' = \{(x_s) \in R^{n+1} \mid x_{\pi(i)} = 0\} \subseteq R_i^{n+1}$$

since $g(X_{\pi(i),\pi(i)}) = 1$: if we have an element in $x \in L_i \cap L_i'$ then it is of the form λv_i and then $\lambda = 0$, so it is zero. Also every element $x \in R^{n+1}$ can be written as

$$x = (x - x_{\pi(i)} \cdot v_i) + (x_{\pi(i)} \cdot v_i) \in L_i' + L_i.$$

Invertibility is also clear since L_i is isomorphic to R given by the map $R \to L_i$ with $\lambda \mapsto \lambda v_i$.
Now we let $R_{i,j}$ be the ring such that $\operatorname{Spec}(R_{i,j}) = \operatorname{Spec}(R_i) \cap \operatorname{Spec}(R_j)$ and then we have that the images of v_i and v_j in $R_{i,j}^{n+1}$ under the maps $\varphi_i : R_i \to R_{i,j}$ and

$\varphi_j : R_j \to R_{i,j}$ differ by multiplication with the unit $g_i(X_{i,j})$ by the definition of projective space. Therefore we find that

$$(\varphi_i)^*(L_i) = (\varphi_j)^*(L_j)$$

as subspaces of $R_{i,j}^{n+1}$. Thus using Lemma 15.17 we get a well-defined one-dimensional complementable subspace of R^{n+1} assembling the L_i.

Conversely assume that we have a 1-dimensional complementable subspace $L \subseteq R^{n+1}$. We want to produce a map $\mathrm{Spec}(R) \to \mathbb{P}^n$ and will construct this locally around each point $x \in \mathrm{Spec}(R)$.
First L is as a quotient of R^{n+1} finitely generated and the vector space $L \otimes_R \kappa(x)$ is invertible, hence 1-dimensional. Now pick an element $v \in L \subseteq R^{n+1}$ whose image generates this vector space. Such an element always exists since we can write any element of $L \otimes_R \kappa(x)$ as $\sum_i l_i \otimes [r_i]/[s_i] = \sum_i l_i r_i \otimes 1/[s_i]$ so that a sufficient multiple (by the product of all the s_i) lies in the image. Therefore we deduce by the Nakayama lemma that upon passing to an open neighborhood $\mathrm{Spec}(R') = D_f \subseteq \mathrm{Spec}(R)$ around x we have that $L[1/f]$ is generated by v. We will assume for simplicity that $R' = R$ since we need to construct our map locally anyway. Thus we can assume that L is generated by a single element $v \in L$. Not all of the coordinates v_i can lie in x since otherwise the induced element in $L \otimes_R \kappa(x)$ would be zero. Thus pick a j such that $v_j \notin x$ and form the ring $R[1/v_j]$ whose spectrum is an open neighborhood of x. Again we replace R be $R[1/v_j]$ and therefore assume that the j-th coordinate of v is invertible in R.
In total we have shown that we can assume that $L = (v) \subseteq R^{n+1}$ for some v with j-th coordinate invertible. We then obtain a map

$$\mathrm{Spec}(R_i) \to \mathrm{Spec}\left(\frac{\mathbb{Z}[X_{0,j}, \dots, X_{n,j}]}{X_{j,j} - 1}\right) \to \mathbb{P}^n$$

which sends $X_{s,j}$ to v_s/v_j. We claim that this map does not depend on the choice of coordinate j. If we have another coordinate j' such that $v_{j'}$ is a unit, then we see that the map

$$\mathrm{Spec}(R_i) \to \mathrm{Spec}\left(\frac{\mathbb{Z}[X_{0,j'}, \dots, X_{n,j'}]}{X_{j',j'} - 1}\right) \to \mathbb{P}^n$$

is the same as the map above as the two maps differ by multiplication with $X_{i,j}$ by definition of the charts of $\mathbb{P}^n$. This also shows on double intersections that the maps agree and glue to a map $\mathrm{Spec}(R) \to \mathbb{P}^n$.

Finally we claim that the two given constructions are inverse to each other. This follows immediately from the definitions, so let us be brief: if we start with a map $g : \mathrm{Spec}(R) \to \mathbb{P}^n$ then we produce a subspace $L \subseteq R^{n+1}$ that is locally (on the open set V_i) spanned by

$$v_i = \big(g_i(X_{0,\pi(i)}), g_i(X_{1,\pi(i)}), \dots, g_i(X_{n,\pi(i)})\big)$$

which then gives back the respective vector. Conversely any 1-dim complementable subspace $L \subseteq R^{n+1}$, locally of the form $L = (v)$ for $v = (v_1, \dots, 1, \dots, v_n)$ leads to the map that sends $X_{s,j}$ to v_i and thus gives back the subspace itself. $\square$

16. Nakayama's Lemma

Definition 16.1. *For a commutative ring R we set*

$$\mathrm{rad}(R) = \bigcap \{\mathfrak{m} \mid \mathfrak{m} \subseteq R \textit{ maximal ideal}\}$$

and call it the Jacobson radical *of R.*

Clearly the Jacobson radical is an ideal (as an intersection of ideals).

Example 16.2. For fields the Jacobson radical is zero. For the integers $\mathbb{Z}$ we have that $\mathrm{rad}(\mathbb{Z}) = \bigcap_{p \text{ prime}}(p) = 0$.
Let R be a local ring with maximal ideal $\mathfrak{m}$. Then $\mathrm{rad}(R) = \mathfrak{m}$. This for example applies to the local rings $A_\mathfrak{p}$ or $\mathcal{O}_x$.

Remark 16.3. We have seen in Theorem 6.11 that the nilradical $\sqrt{0}$ can be written as the intersection of all prime ideals. Since every maximal ideal is prime we deduce that we have an inclusion

$$\sqrt{0} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} \subseteq \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m} = \mathrm{rad}(R) \ .$$

The Jacobson radical corresponds under the corresponce of radically closed ideals and closed sets to the closure of the set of closed points in $\mathrm{Spec}(R)$.

Proposition 16.4. *We have that*

$$\mathrm{rad}(R) = \{x \in R \mid 1 - xy \in R^\times \textit{ for all } y \in R\}$$

Proof. Assume that $x \in \mathrm{rad}(R)$ and suppose $1 - xy$ is not a unit for some y. Then $1 - xy \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. But since $xy \in \mathrm{rad}(R) \subseteq \mathfrak{m}$ we get that $1 \in \mathfrak{m}$ and a contradiction.
Assume conversely that $1 - xy$ is a unit for all y and $x \notin \mathrm{rad}(R)$. Then for some maximal ideal we have $x \notin \mathfrak{m}$, thus $1 = yx + m$ for some $m \in \mathfrak{m}, y \in R$ (otherwise $(x) + \mathfrak{m}$ would be a larger proper ideal). This implies that $m = 1 - yx$ is a unit which is a contradiction. □

Example 16.5. From this formula we can again see that all nilpotent elements x are contained in the Jacobson radical, since in this case an inverse to $(1 - yx)$ is given by the geometric series

$$1 + yx + y^2x^2 + ...$$

which is a finite sum due to nilpotency of x.

Theorem 16.6 (Nakayama's Lemma, Abstract Version). *Assume that M is a finitely generated R module and $I \subseteq \mathrm{rad}(R)$ an ideal such that $M \otimes_R R/I = 0$. Then $M = 0$.*

Note that we have $M \otimes_R R/I = \mathrm{coker}(M \otimes_R I \to M \otimes_R R) = M/IM$. Thus the assumption is equivalent to the fact that the inclusion

$$IM = \{im \mid i \in I, m \in M\} \subseteq M$$

is an equality.

Proof. Let $m_1, ..., m_n$ be a minimal generating set of M. If M is non-trivial then we have $n \geq 1$ and $m_n \neq 0$. By assumption $m_n \in IM$ so that we have

$$m_n = \sum_{k=1}^{n} i_k m_k$$

hence

$$(1 - i_n)m_n = \sum_{k=1}^{n-1} i_k m_k \in (m_1, ..., m_{n-1}) \ .$$

Since $i_n \in I \subseteq \mathrm{rad}(R)$ we have that $(1 - i_n)$ is a unit in contradiction to the minimality of the generating set. □

Remark 16.7. The assertion of the Nakayama Lemma fails completely for non-finitely generated modules. For example we can take $R = \mathbb{Z}_{(2)}$ and $I = (2)$ and the $\mathbb{Z}_{(2)}$ module to be $\mathbb{Q}$. Then certainly $\mathbb{Q}/2 = 0$ but $\mathbb{Q}$ is non-zero.

Corollary 16.8. *Assume that M is a finitely generated R-module and $I \subseteq \mathrm{rad}(R)$. Assume that some elements $m_1, ..., m_n$ are such that the images are generators for M/I as a R/I-module. Then the m_i also generate M.*

Proof. Consider the submodule $M' \subseteq M$ generated by $m_1, ..., m_n$ and form the quotient M/M'. Then we have that $(M/M') \otimes_R R/I = M/(M' + IM) = 0$ and thus $M/M' = 0$ which shows that $M' = M$. □

We are now ready to prove the statement we have used as Lemma 16.9 above:

Corollary 16.9 (Nakayama's Lemma, prime ideal version). *Assume that M is a finitely generated R-module with elements $m_1, ..., m_n \in M$ such that for some prime ideal $x \in \mathrm{Spec}(R)$ the images of the elements $m_1, ..., m_n$ generate $M \otimes_R \kappa(x)$ as a vector space over $\kappa(x)$. Then there is some $f \in R$ with $f \notin x$ such that the elements $m_1, ..., m_n$ already generate $M[1/f]$ as a module over $R[1/f]$.*

Proof. Consider the local ring R_x. The maximal ideal is $\mathfrak{m} = xR_x$ and $R_x/\mathfrak{m} = \kappa(x)$. Thus if we form the R_x-module $M' := M \otimes_R R_x$ we find that

$$M'/\mathfrak{m} = M \otimes_R \kappa(x) \ .$$

Thus the images of $m_1, .., m_n$ generate M' by Corollary 16.8 above. We have that $R_x = \mathrm{colim}_{x \in D_f} R[1/f]$ (see Example 9.4) so that

$$M' = M \otimes_R R_x = \mathrm{colim}_{x \in D_f} M[1/f] \ .$$

This is a filtered colimit. Thus if we pick generators $y_1, ..., y_k$ of M they can all be written as linear combinations of the m_i in M', thus in some $M[1/f]$ for f large enough (this also easily follows by thinking of elements in $M[1/f]$ as fractions). This f now does the job. □

17. Quasi-coherent sheaves

Recall that schemes are a global geometric version of rings. Similarly we'll introduce a geometric generalization of modules. Recall that for an R-module M and an element $f \in R$ we can functorially form

$$M[1/f] = M \otimes_R R[1/f] \ .$$

which is an $R[1/f]$-module (and will sometimes also be considered as an R-module).

Proposition 17.1. *For any R-module M we have a sheaf of abelian groups $\widetilde{M} \in$ $\mathrm{Shv}(\mathrm{Spec}(R); \mathrm{Ab})$ which is given on principal opens by the underlying abelian group of*

$$D(f) \mapsto M[1/f] = M \otimes_R R[1/f] \ .$$

PROOF. We have to show functoriality and the sheaf property. Functoriality is clear since $M[1/f]$ only depends on $D(f)$ i.e. the radical ideal generated by (f) and then we simply look at inclusions of open sets. For the sheaf property we have to assume that we have elements $(f_1, .., f_n)$ such that $1 \in (f_1, ..., f_n)$ and then show that the diagram

$$M \to \prod M[1/f_i] \rightrightarrows \prod M[1/f_i, 1/f_j]$$

is an equalizer.This works exactly as in the proof of Theorem 8.11 for the structure sheaf, details omitted. □

The stalks of $\widetilde{M}$ are given by

$$\widetilde{M}_x = \operatorname{colim}_{x \in D_f} M[1/f] = M[(R \setminus x)^{-1}] = M_x$$

as one immediately sees from the definition.

DEFINITION 17.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{M}$ together with a morphism of sheaves*

$$\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$$

such that for every U the induced morphism $\mathcal{O}(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$ makes $\mathcal{M}(U)$ into a $\mathcal{O}_X(U)$-module. Morphisms between sheaves $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{O}_X$-modules are sheaf morphisms that commute with this action and are denotes as

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \ .$$

We write the category of sheaves of $\mathcal{O}_X$-modules also as $\operatorname{Mod}(\mathcal{O}_X)$.

Note that by definition this means that we want to give for every U the abelian group $\mathcal{M}(U)$ the structure of a $\mathcal{O}_X(U)$-module compatible with the restriction maps.

EXAMPLE 17.3. This is an example for people who know what a vector bundle is. So assume X is a topological space and $p : \mathcal{V} \to X$ is a continuous real vector bundle. In particular $\mathcal{V}$ is itself a topological space and for each $x \in X$ we have the structure of an $\mathbb{R}$-vector space on $\mathcal{V}_x := p^{-1}(x)$. We consider the sheaf of sections of $\mathcal{V}$ defined as

$$U \subseteq X \mapsto \{s : U \to \mathcal{V}|_U \mid ps = \mathrm{id}\} \ .$$

This is a sheaf of C^0_X-modules by pointwise multiplication. In fact, one can completely recover vector bundles up to isomorphism from this associated sheaf, but we will not go into this here.

PROPOSITION 17.4. *For any R-module M the sheaf $\widetilde{M}$ is canonically a sheaf of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules.*

PROOF. As usual by Lemma 8.10 it suffices to define everything on principal opens. The action is given on $U = D_f$ by

$$R[1/f] \times M[1/f] \to M[1/f]$$

and is natural (in fact, $M[1/f] = M \otimes_R R[1/f]$). The rest is clear. □

We now prove an analogue of Theorem 12.2 about being able to recover affine schemes by its global sections. To this end note that for the sheaf of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules $\widetilde{M}$ we can recover the R-module M as global sections (i.e. evaluation at $\operatorname{Spec}(R)$).

Proposition 17.5. *Let M be an R-module for some ring R and $\mathcal{N}$ be a sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules on* $\mathrm{Spec}(R)$. *Then the map*

$$\Gamma : \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(R)}}(\widetilde{M}, \mathcal{N}) \to \mathrm{Hom}_R(M, \mathcal{N}(\mathrm{Spec}(R))$$

is a bijection.

Proof. We first prove injectivity. Assume that we have two maps $\varphi, \psi : \widetilde{M} \to \mathcal{N}$ that agree on global sections. We will show that they agree on all principal opens $D(f) \subseteq \mathrm{Spec}(R)$ which is enough (another application of Lemma 8.10). But by definition we have a commutative square

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\varphi=\psi"] & M[1/f] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\psi"] \\
\mathcal{N}(\mathrm{Spec}(R)) \arrow[r] & \mathcal{N}(\mathrm{Spec}(R[1/f]))
\end{tikzcd}$$

where the right hand maps are maps of $R[1/f]$-modules and the left hand map is a map of R-modules. What we use now is that $M[1/f] = M \otimes_R R[1/f]$ which implies that for any $R[1/f]$-module N (here $N = \mathcal{N}(\mathrm{Spec}(R[1/f])))$ the induced map

$$\mathrm{Hom}_{R[1/f]}(M[1/f], N) \to \mathrm{Hom}_R(M, N) \tag{9}$$

is a bijection (this is a special case of Exercise 1 and 4 of sheet 11). That is the right vertical maps in the diagram above are completely determined by the counter-clockwise composition and therefore have to agree. This shows injectivity. For surjectivity, given a map $\varphi : M \to \mathcal{N}(\mathrm{Spec}(R))$, we would like to extend it for each $f \in R$ as such

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\varphi"] & M[1/f] \arrow[d, dashed] \\
\mathcal{N}(\mathrm{Spec}(R)) \arrow[r] & \mathcal{N}(\mathrm{Spec}(R[1/f]))
\end{tikzcd}$$

in a natural way where the dashed arrow is $R[1/f]$-linear. This is again automatic by the universal property (9). The dashed arrows for varying f are then clearly natural and induce a map of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules. □

Corollary 17.6. *The functor*

$$\mathrm{Mod}_R \to \mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}(R)}) \qquad M \mapsto \widetilde{M}$$

from R-modules to sheaves of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is fully faithful and left adjoint to the global sections functor.

The question now is to identify the essential image of this functor and to 'globalize' it.

Definition 17.7. *Let X be a scheme. A sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules on X is called* quasi-coherent *if there exists an open cover $U_i = \mathrm{Spec}(A_i)$ of X by affines such that $\mathcal{M}|_{U_i}$ is as a sheaf of $\mathcal{O}_{U_i}$-modules isomorphic to $\widetilde{M_i}$ for some A_i-module M_i. We denote the category of quasi-coherent sheaves by*

$$\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(\mathcal{O}_X)$$

which is defined as a full subcategory of $\mathcal{O}_X$-modules.

Example 17.8. For any R-module M the sheaf $\widetilde{M} \in \mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}(R)})$ is quasi-coherent. We can simply take the trivial open cover for this. In fact, we could take any other open cover by principal opens since we have by definition

$$(\widetilde{M})|_{D_f} \cong \widetilde{M[1/f]}$$

as sheaves of $\mathcal{O}_{\mathrm{Spec}(R[1/f])}$-modules, where $M[1/f]$ is considered as a $R[1/f]$-module.

Proposition 17.9. *Taking global sections induces an equivalence of categories*

$$\mathrm{QCoh}(\mathrm{Spec}(R)) \xrightarrow{\simeq} \mathrm{Mod}_R$$

with inverse given by $M \mapsto \widetilde{M}$.

Proof. We already know that $\mathrm{Mod}_R \to \mathrm{QCoh}(\mathrm{Spec}(R))$ is fully faithful by what we have done so far. Thus it only remains to show that a quasi-coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\mathcal{M}$ is always of the form $\widetilde{M}$ for $M = \mathcal{M}(\mathrm{Spec}(R))$ its global sections. By the adjunction of Corollary 17.6 we get a morphism

$$\widetilde{M} \to \mathcal{M}$$

of sheaves of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules which we have to show is an isomorphism (this is the counit of the adjunction). It suffices to show that for each $D(g) \subseteq \mathrm{Spec}(R)$ the induced map

$$M[1/g] = \widetilde{M}(D(g)) \to \mathcal{M}(D(g))$$

is an equivalence.
Consider an open cover $(U_i)_{i \in I}$ as in the definition of quasi-coherence. By passing to smaller opens U_i we can assume that $U_i = D(f_i)$ is principal. Since $\mathrm{Spec}(R)$ is quasi-compact we can moreover assume that I is finite. Then we get that $\mathcal{M}(U_i) = M_i$ and $\mathcal{M}(U_i \cap U_j) = M_i[1/f_j] = M_j[1/f_i] =: M_{i,j}$. By the sheaf property for $\mathcal{M}$ we can write M as the equalizer

$$\mathrm{Eq}\left(\prod_i M_i \rightrightarrows \prod_{i,j} M_{i,j}\right)$$

Now localization is a filtered colimit, so commutes with equalizers (this is the fact that filtered colimits of abelian groups/sets commute with kernels/equalizers). It also commutes with finite products since these are also finite coproducts and colimits commute with colimits. Thus we get that

$$\begin{aligned} M[1/g] &\cong \mathrm{Eq}\left(\prod_i M_i[1/g] \rightrightarrows \prod_{i,j} M_{i,j}[1/g]\right) \\ &\cong \mathrm{Eq}\left(\prod_i \mathcal{M}(D(gf_i)) \rightrightarrows \prod_{i,j} \mathcal{M}(D(gf_if_j))\right) \\ &\cong \mathcal{M}(D(g)) \end{aligned}$$

where for the last equality we have used the sheaf property for $\mathcal{M}$ and the fact that $D(gf_i)$ is a cover of $D(g)$ with intersections $D(f_ig)$. This shows the claim (the induced map is this isomorphism as one sees unwinding the constructions). $\square$

18. Descent for modules

Let X be a topological space, $\mathcal{C}$ be a category with all small limits and $F \in \mathrm{Shv}(X;\mathcal{C})$ be a sheaf. For any open cover $(U_i)_{i\in I}$ of X we get restricted sheaves

$$F_i := F|_{U_i} \in \mathrm{Shv}(U_i;\mathcal{C})$$

together with isomorphisms

$$\varphi_{ij} : F_i|_{U_i\cap U_j} \cong F|_{U_i\cap U_j} \cong F_j|_{U_i\cap U_j}$$

in $\mathrm{Shv}(U_i \cap U_j;\mathcal{C})$. These isomorphisms satsify on triple intersections $U_{ijk} := U_i \cap U_j \cap U_j$ the condition that

$$(\varphi_{jk}|_{U_{ijk}}) \circ (\varphi_{ij}|_{U_{ijk}}) = \varphi_{ik}|_{U_{ijk}}$$

as maps $F_i|_{U_{ijk}} \to F_j|_{U_{ijk}}$. We will also abusively write this as $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ and leave the restrictions implicit. We refer to this condition as the *cocycle condition.*

Definition 18.1. *In the situation of a topological space X and an open cover $\mathcal{U} = (U_i)_{i\in I}$ we define the* descent category of sheaves

$$\mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C}))$$

to be the category with objects given by descent data*, that is families $(F_i \in \mathrm{Shv}(U_i;\mathcal{C}))_{i\in I}$ together with isomorphisms $\varphi_{ij} : F_i|_{U_{ij}} \to F_j|_{U_{ij}}$ in $\mathrm{Shv}(U_{ij};\mathcal{C})$ that satisfy the cocycle condition on triple intersections. A morphism (F_i, φ_{ij}) to (G_i, ψ_{ij}) is given by a family of morphisms $f_i : F_i \to G_j$ such that $f_j \circ \varphi_{ij} = \psi_{ij} \circ f_i$.*

It is clear that this defines a category under composition of morphisms. Moreover we have a canonical functor

$$\mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C}))$$

which sends F to the descent datum with $F_i := F|_{U_i}$ as described above.

Proposition 18.2 (Descent for sheaves). *The functor*

$$\mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C}))$$

is an equivalence of categories.

Sketch. We want to construct an inverse functor. To this extend, we define a basis $\mathcal{B}$ of the topology of X to be given by those open sets $U \subseteq X$ which are contained in one of the U_i. This basis is closed under intersection, so that Lemma 8.10 applies and we have that in the composition

$$\mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}_{\mathcal{B}}(X;\mathcal{C}) \to \mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C}))$$

the first functor is an equivalence. Thus it suffices to show that the second functor is an equivalence. We will do this by giving an explicit inverse. To this extend we start with a descent datum (F_i, φ_{ij}). Then we want to construct a functor

$$F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\ .$$

For $U \subseteq \mathcal{B}$ we have $U \subseteq U_i$ and thus can set

$$F(U) := F_i(U).$$

This is well-defined up to canonical isomorphism: if we have also $U \subseteq U_j$ then we have that $F_i(U) \cong F_j(U)$ be the isomorphism φ_{ij}. Moreover F defines a sheaf and the restriction of F to each U_i is isomorphic to F_i (these arguments are a bit sketchy and should be proven a bit more carefully). $\square$

We have a similar statement for $\mathcal{O}_X$-modules and quasi-coherent sheaves. The descent categories are defined in the obvious way and we have similar functors as before. The result is the following:

Proposition 18.3. *Let X be a scheme with an open cover $(U_i)_{i\in I}$. Then the functors*

$$\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Desc}_{\mathcal{U}}(\mathrm{Mod}(\mathcal{O}_-))$$

and

$$\mathrm{QCoh}(X) \to \mathrm{Desc}_{\mathcal{U}}(\mathrm{QCoh}(-))$$

are equivalences of categories.

Proof. The second statement follows from the first since quasi-coherent sheaves are a full subcategory of $\mathcal{O}_X$-modules defined by a local condition. For the first we simply observe that we can use the inverse functor on the level of sheaves (as constructed in the proof of Proposition 18.2) and equip everything canonically with $\mathcal{O}_X$-module structures. □

We finally specify the last statement to the case of affine schemes. Note that the restriction functor

$$-|_U : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$$

for $X = \mathrm{Spec}(R)$ and $U = \mathrm{Spec}(R[1/f])$ principal open is under the equivalences

$$\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R \qquad \text{and} \qquad \mathrm{QCoh}(U) \simeq \mathrm{Mod}_{R[1/f]})$$

given by the functor

$$M \mapsto M[1/f] = M \otimes_R R[1/f] \ .$$

Thus we get the following statement for modules, which is not so easy to prove directly (without using the formalism of sheaves);

Corollary 18.4 (Descent for modules). *Let R be a ring with a principal open cover $(R_i = R[f_i^{-1}])_{i\in I}$ (i.e. $\mathrm{Spec}(R_i)$ an open cover of $\mathrm{Spec}(R)$). Then the functor*

$$\begin{aligned}\mathrm{Mod}_R &\to \mathrm{Desc}_{(R_i)}(\mathrm{Mod}) \\ &= \{M_i \in \mathrm{Mod}_{R_i}, \quad \varphi_{ij} : M_i[f_j^{-1}] \xrightarrow{\simeq} M_j[f_i^{-1}] \mid \textit{cocycle condition}\}\end{aligned}$$

is an equivalence of categories.

This equivalence shows that we can construct and manipulate modules 'locally' similar to functions. This is an important principle. When we have a family $M_i \in \mathrm{Mod}_{R_i}$ of locally defined modules with transition functions φ_{ij} then we also say that the corresponding module M is 'glued' from the M_i.
Note that under this equivalence the tensor product $M \otimes_R N$ corresponds to the operations of tensoring $M_i \otimes_{R_i} N_i$ and forming the corresponding isomorphisms $\varphi_{ij}^M \otimes \varphi_{ij}^M$.

Lemma 18.5. *An R-module M is invertible precisely if for a principal cover $(R_i)_{i\in I}$ as before the induced modules $M_i := M[f_i^{-1}] = M \otimes_R R_i$ are invertible over R_i.*

Proof. One direction is immediate from Example 15.8. So assume that M is such that the M_i are invertible with inverses L_i. Then we want to 'glue' the L_i to a global module L. To this extend we note that $L_i|_{U_{ij}}$ is inverse to M_{ij}. Inverse modules are unique in the sense that if N_1 and N_2 are inverse to the same module M_{ij}, then we get an induced isomorphism $N_1 \to N_2$ since:

$$N_1 \cong N_1 \otimes M_{ij} \otimes N_2 \cong N_2 \ .$$

Applying this observation to $L_i|_{U_{ij}}$ and $L_j|_{U_{ij}}$ we get isomorphisms between them and it is not so hard to check that they satisfy the cocycle identity. Therefore we can 'glue' them to get a global module L over R. This now has the property that $M \otimes_R L$ is equivalent to the unit R. This follows since it is true on U_i for each $i \in I$ and the resulting glueing functions are on double intersections given by the identities as one easily verifies. $\square$

Finally we can prove the last open part of our identification of the functor of points of $\mathbb{P}^1$, which was Lemma 15.17:

The functor

$$R \mapsto \{\text{One dimensional complementable subspaces } L \subseteq R^{n+1}\}$$

satisfies descent.

Proof. Assume that we have a principal cover $\big(R_i = R[f_i^{-1}]\big)_{i\in I}$ of R with maps $\varphi_j : R_i \to R_{ij}$ (as usual the R_{ij} are the double intersections). For a family of complementable, 1-dimensional subspaces $L_i \subseteq R_i^{n+1}$ such that

$$(\varphi_j)^*(L_i) = (\varphi_i)^*(L_j)$$

we need to find a unique $L \subseteq R^{n+1}$ which base-changes to the L_i.
In order to do this we need to supply a descent datum for L. Thus we choose the local modules to be the L_i's and then the isomorphism $\varphi_{ij} : L_i[f_j^{-1}] \to L_j[f_i^{-1}]$ is given be the isomorphism

$$L_i[f_j^{-1}] \cong L_i \otimes_{R_i} R_i[f_j^{-1}] \cong (\varphi_j)^*(L_i) = (\varphi_i)^*(L_j) \cong L_j[f_i^{-1}].$$

These isomorphisms then satisfies the cocycle condition and we can 'glue' them to get a module L. By the previous Lemma 18.5 this is an invertible module.
Similarly we also glue the inclusion morphisms $L_i \to R_i^{n+1}$ to get a morphisms $L \to R^{n+1}$. Now we need to verify that the map $L \to R^{n+1}$ is a split inclusion. We skip this part here for time reasons. [6] $\square$

19. Line bundles

Definition 19.1. *Let X be a scheme and $\mathcal{M}, \mathcal{N}$ be $\mathcal{O}_X$-module. Then we define $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ as the sheafification of*

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U) \ .$$

This is again an $\mathcal{O}_X$-module sheaf.

Proposition 19.2. *For R-modules M and N we have*

$$\widetilde{M} \otimes_{\mathcal{O}_{\mathrm{Spec}(R)}} \widetilde{N} \cong \widetilde{M \otimes_R N} \ .$$

If X is a scheme and M and N are quasi-coherent. Then so is $M \otimes_{\mathcal{O}_X} N$.

[6] Sketch: dualise the situation, then one has to show that being surjective between quasi-coherent sheaves is a local property on affines. This folliows by looking at the cokernel and noting that the cokernel of a map between quasi-coherent sheaves is again quasi-coherent, so vanishes precisely if the global sections are trivial.

Proof. For $U = D(f) \subseteq \operatorname{Spec}(R)$ we have

$$\begin{aligned}\widetilde{M}(U) \otimes_{\mathcal{O}_{\operatorname{Spec}(R)}(U)} \widetilde{N}(U) &= M[1/f] \otimes_{R[1/f]} N[1/f] \\ &\cong (M \otimes_R N)[1/f] \\ &\cong (\widetilde{M \otimes_R N})(U) \ .\end{aligned}$$

This is a sheaf, so there is no sheafification necessary on principal opens.
The second statement can be checked locally (by the definition of quasi-coherent sheaves). But locally it follows from the previous assertion (see the discussion in Example 11.19 for why we can argue locally). □

So we see that the tensor product of $\mathcal{O}_X$-modules is the global geometric version of the tensor product of modules. Finally we introduce the global version of invertible sheaves.

Definition 19.3. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{L}$ is called invertible or a line bundle over X, if there exists an $\mathcal{O}_X$-module $\mathcal{L}'$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \cong \mathcal{O}_X$. We also write $\mathcal{L}'$ as $\mathcal{L}^{-1}$ and call it the inverse of $\mathcal{L}$.*

Note that the inverse is unique (up to isomorphism) by the usual argument (see the proof of Lemma 18.5) and therefore well-defined. An example of a line bundle is $\mathcal{O}_X$ considered as a $\mathcal{O}_X$-module by left multiplication. We refer this as the trivial line bundle.

Proposition 19.4.
(1) *Every invertible $\mathcal{O}_X$-module is quasi-coherent.*
(2) *If $X = \operatorname{Spec}(R)$ is affine then invertible $\mathcal{O}_X$-modules are via taking global sections equivalent to invertible R-modules.*
(3) *Given an open cover of X. Then an $\mathcal{O}_X$-module $\mathcal{M}$ is invertible precisely if it restricts to invertible modules on all of the open sets.*
(4) *For any line bundle $\mathcal{L}$ over X there exists an open cover such that the restriction to this open cover is trivial.*

We skip this proof for time reasons, but note that the third statement can also be interpreted in terms of descent, namely that the assignment

$$X \mapsto \{\text{line bundles over } X\}$$

satisfies descent similar to Proposiiton 18.3. Combining this with the fourth statement one could also define line bundles to be $\mathcal{O}_X$-modules which are locally isomorphic to $\mathcal{O}_X$. Note that this is even in the affine case a highly non-trivial statement, namely that each invertible R-module M is locally trivial.
In total we have now introduced a number of global geometric statements:

algebraic object	global geometric object
Ring	Scheme
Module	quasi-coherent sheaf
$- \otimes_R -$	$- \otimes_{\mathcal{O}_X} -$
invertible module	line bundle

We will continue this table next semester.

Theorem 19.5. *For any scheme X we have a natural isomorphism*

$$\mathbb{P}^n(X) := \operatorname{Hom}_{\mathrm{Sch}}(X, \mathbb{P}^n) \cong \{\mathcal{L} \subseteq \mathcal{O}_X^{n+1} \mid \mathcal{L} \text{ invertible, locally complementable}\}$$

Here the inclusion means that $\mathcal{L}$ is a subsheaf of $\mathcal{O}_X^{n+1}$ as an $\mathcal{O}_X$-module sheaf, that is for each U we have that $\mathcal{L}(U) \subseteq \mathcal{O}_X^{n+1}(U)$ and that this subset is closed under restriction.

Again we have to skip the proof. As a result of the theorem we can consider the identity $\mathrm{id} : \mathbb{P}^n \to \mathbb{P}^n$. This gives rise to an invertible $\mathcal{O}_X$-module $\mathcal{L} \subseteq \mathcal{O}_X^{n+1}$.

DEFINITION 19.6. *We define line bundles $\mathcal{O}_{\mathbb{P}^n}(n)$ for $n \in \mathbb{Z}$ over $\mathbb{P}^n$ by setting*

$$\mathcal{O}_{\mathbb{P}^n}(-n) := \mathcal{L}^{\otimes n} = \mathcal{L} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{L}$$
$$\mathcal{O}_{\mathbb{P}^n}(n) := \mathcal{L}^{\otimes -n} = \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \ .$$

for $n \geq 0$ (which is to say that $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$.

REMARK 19.7. It turns out that that the line bundles $\mathcal{O}_{\mathbb{P}}(n)$ with $n \in \mathbb{Z}$ are all invertible modules over $\mathbb{P}^n$ (and are mutually non-isomorphic).

EXAMPLE 19.8. Consider the scheme $\mathbb{P}^1$. Then we have the open cover U_0 and U_1 with interection $U_0 \cap U_1 = \mathrm{Spec}(\mathbb{Z}[x^{\pm}])$. Then an invertible module is given by a triple

- $\mathcal{L}_0$ line bundle over U_0 ;
- $\mathcal{L}_1$ line bundle over U_1 ;
- An isomorphism $L_0|_{U_0 \cap U_1} \to L_1|_{U_0 \cap U_1}$.

Note that this is really everything, since there are no-triple intersections so the cocycle condition is trivial (except it requires that the iso on $U_1 \cap U_0$ is ihe inverse). Examples are given by $L_0 = \mathcal{O}_{U_0}$ and $L_1 = \mathcal{O}_{U_1}$. Then the isomorphism is given by an isomorphism $\mathbb{Z}[x^{\pm}] \to \mathbb{Z}[x^{\pm}]$ of $\mathbb{Z}[x^{\pm}]$-modules or equivalently a unit in $\mathbb{Z}[x^{\pm}]$ (by which we multiply to get the map). The units are precisely given by $\pm x^n$ for $n \in \mathbb{Z}$ and it is easy to see that the sign does not matter for the isomorphism class of the glued line bundle. The bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ is under this isomorphism given by the unit x^n.

CHAPTER 2

Algebraic Geometry II, Sommersemester 2022

This course was taught in Sommersemester 2022 in Münster. The goal is to continue the theory of schemes. We will talk about the Picard group first and then cover properties of morphisms like affine, separated and proper. We will also introduce the dimension and projective schemes. Then we will study curves and how they are associated to field extension, and why separated curves are quasi-projective.

(1) R. Hartshorne: Algebraic Geometry GTM 52. Springer.
(2) D. Mumford: The red book of varieties and schemes. Springer LN 1358.
(3) U. Goertz, T. Wedhorn: Algebraic Geometry I. Vieweg.
(4) A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique.
(5) P. Scholze: Algebraic Geometry I lecture notes (typed by Jack Davies)

The lectures are Monday 2-4 and Thursday 2-4 in M6. The exercises are friday 10-12 and will be held by Achim Krause.

1. Summary and Preliminaries

We first want to recall the content of the first lecture and what we need to know to follow this lecture. In particular we will review Grothendieck's notion of schemes which is central to everything. Everything that we outline here is covered in detail in the first part of the script.

First we assume that the reader is familiar with the notion of sheaves: a sheaf on a topological space X is a functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ which satisfies descent, that is for every collection of open sets $\{U_i \subseteq X\}_{i \in I}$ the induced diagram

$$F(\cup_i U_i) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer in $\mathcal{C}$. Typically we will use this for $\mathcal{C}$ being rings or abelian groups in which case it can be translated into a statement about elements and it becomes appearent that this is a local-to-global principle. For every point $x \in X$ we can the define the stalk

$$F_x := \mathrm{colim}_{x \in U} F(U) \ .$$

Definition 1.1.
- *A locally ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of commutative rings $\mathcal{O}_X$ on X which has the property that the stalks $\mathcal{O}_{X,x}$ are local rings (i.e. have a unique maximal ideal) are local rings.*
- *A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a morphism $f : X \to Y$ and a morphism $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves over X such that for every $x \in X$ the induced morphism*
$$f^\sharp_x : \mathcal{O}_{Y,f(x)} = (f^{-1}\mathcal{O}_Y)_x \to \mathcal{O}_{X,x}$$
is a morphism of local rings, i.e. the preimage of the maximal ideal is the maximal ideal.

Example 1.2. Let R be a commutative ring. We define a locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ as follows: the points of $\mathrm{Spec}(R)$ are given by prime ideals in R. The topology is the Zariski topology: for an arbitrary subset $M \subseteq R$ we declare the vanishing locus

$$V(M) = \{x \in \mathrm{Spec}(R) \mid M \subseteq x\} = \{x \in \mathrm{Spec}(R) \mid f(x) = 0 \,\forall\, f \in M\}$$

to be closed and consequently the complements to be open. In particular for a given element $f \in R$ we have the principal open set

$$D(f) = V(f)^c = \{x \in \mathrm{Spec}(R) \mid f \notin x\} = \{x \in \mathrm{Spec}(R) \mid f(x) \neq 0\} \ .$$

There is a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which has the property that

$$\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R[1/f] \ .$$

In particular the global sections are given by R. So geometrically we can think of R as functions on this geometric object. It turns out that the stalks are given by

$$\mathcal{O}_{\mathrm{Spec}(R),x} = R_x$$

(the localization at x) which are local rings. Thus the pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a locally ringed space. We will most of the time denote this locally ringed space simply by $\mathrm{Spec}(R)$ and leave the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ implicit.

Definition 1.3. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to* $\mathrm{Spec}(R)$*, that is for every point $x \in X$ there exists and open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic as a locally ringed space to* $\mathrm{Spec}(R)$*.*

Theorem 1.4. *The assignment $R \mapsto \mathrm{Spec}(R)$ induces a fully faithful functor from the opposite of the category of rings to the category of schemes. Objects in the essential image (that is schemes isomorphic to* $\mathrm{Spec}(R)$*) are called affine schemes. This functor has a left adjoint given by taking global sections.*

Example 1.5. Assume that we have a ring k (e.g. a field) and a family of polynomials $p_1, ..., p_k \in k[X_1, ..., X_n]$. Then we have the ring $A = k[X_1, ..., X_n]/(p_1, ..., p_k)$ and the (affine) scheme $\mathrm{Spec}(R)$. The fact that A is k-algebra translates into a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ so that $\mathrm{Spec}(A)$ is canonically a scheme over $\mathrm{Spec}(k)$.
The intuition is that this is the 'geometric object' formed by the zero-set of the polynomials $p_1, ..., p_k$. Let us explain this intuition a bit more, which is central to algebraic geometry. First, as a topological space we have that $\mathrm{Spec}(A)$ is a closed subset of $\mathrm{Spec}(k[X_1, ..., X_n])$ given by the vanishing locus of the p_i. If k is an algebraically closed field, then the closed points of $\mathrm{Spec}(A)$ (which correspond to maximal ideals in R) are given by those elements $x = (x_1, ..., x_n) \in k^n$ such that $p_1(x) = ... = p_k(x) = 0$. But $\mathrm{Spec}(A)$ also has some non-closed points: these correspond to generic points of irreducible closed subsets of k^n (the latter is identified with closed points of $\mathrm{Spec}(k[X_1, ..., X_n])$.

In particular we have the scheme $\mathrm{Spec}(k[x_1, .., x_n])$ and will denote it by $\mathbb{A}^n_k$.

Example 1.6. For every ring k we have a scheme $\mathbb{P}^n_k$ which has an open cover $U_0, ..., U_n \subseteq \mathbb{P}^n_k$ such that

$$U_i \cong \mathrm{Spec}\, k[X_0, ..., \hat{X}_i, ..., X_n] \cong \mathbb{A}^n_k$$

and such that double intersections are given by

$$U_i \cap U_i = \mathrm{Spec}\, k[X_0, ..., X_j^{\pm}, ..., \hat{X}_i, ..., X_n] \ .$$

THEOREM 1.7. *The category of schemes has all finite limits. For a diagram*

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(C) \end{tikzcd}$$

the fibre product is $\mathrm{Spec}(A \otimes_C B)$*.*

In particular products $\mathrm{Spec}(A) \times \mathrm{Spec}(B)$ are given by $\mathrm{Spec}(A \otimes B)$ and general products and fibre products of schemes can be understood by observing that one can locally reduce to the affine case. In particular we have that

$$\mathbb{A}^n_k = \mathbb{A}^n_{\mathbb{Z}} \times \mathrm{Spec}(k) \qquad \mathbb{P}^n_k = \mathbb{P}^n_{\mathbb{Z}} \times \mathrm{Spec}(k) \ .$$

DEFINITION 1.8. *For any scheme X we have the functor of points:*

$$X(-) : \mathrm{Ring} \to \mathrm{Set}$$

given by $X(R) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), X)$*. We call $X(R)$ the R-valued points of X. Sometimes we even call* $\mathrm{Hom}_{\mathrm{Sch}}(S, X)$ *for a scheme S the S-valued points of X and write it as $X(S)$. If X is a scheme over k (i.e over* $\mathrm{Spec}(k)$*) then we similarly can define a functor of points* $\mathrm{Alg}_k \to \mathrm{Set}$ *which we call the relative functor of points.*

THEOREM 1.9. *The assignment* $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ *is fully faithful, in particular every scheme can be recovered by its functor of points. Every object in the essential image satisfies descent.*

One can in fact use this perspective to give a different definition of schemes: functors $\mathrm{Ring} \to \mathrm{Set}$ which are locally isomorphic to $\mathrm{Spec}(R)(-) = \mathrm{Hom}_{\mathrm{Ring}}(R, -)$, but defining what is meant by 'locally' in this setting requires some work and we shall not need this for now.

EXAMPLE 1.10. Consider the scheme $X = \mathbb{A}^n_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x_1, ..., x_n])$. Then we have that

$$X(R) = R^n \ .$$

For $X = \mathbb{A}^n_k$ we have that for every ring R we have that the absolute functor of points is given by

$$X(R) = \mathrm{Hom}_{\mathrm{Ring}}(k, R) \times R^n$$

The functor of points relative to k has a slightly easier form (which one?).

EXAMPLE 1.11. Assume that we have $A = \mathbb{Z}[x_1, ..., x_n]/p_1, ..., p_k$ and $X = \mathrm{Spec}(A)$ as in Example 1.5. Then we find that for any ring R

$$X(R) = \{x \in R^n \mid p_1(x) = ... = p_k(x) = 0\},$$

thus solutions to the equations given by $p_1, ..., p_k$ in the ring R. Clearly this functor 'knows' much more than just solutions over $R = \mathbb{Z}$, which might not even exist, e.g. for something for polynomials like $x^2 - 2$ or $x^2 + 1$. So in some sense the philosphy to understrand a bunch of equations is to record solutions over each ring R.

THEOREM 1.12. *For projective space* $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}}$ *we have that*

$$\mathbb{P}^n(R) \cong \{\textit{One dimensional complementable subspaces of } R^{n+1}\} \ .$$

Finally we have talked about quasi-coherent sheaves on a scheme which is the 'global' version of modules (similar to the way schemes are the global version of rings). To this end we recall for a scheme X the notion of an $\mathcal{O}_X$-module sheaf $\mathcal{M}$, which is a sheaf of abelian groups $\mathcal{M}$ on X which comes equipped with an action of $\mathcal{O}_X$ on $\mathcal{M}$ (in the appropriate sense).

Example 1.13. For any R-module M we have a $\mathcal{O}_{\mathrm{Spec}(R)}$-module sheaf $\widetilde{M}$ which is given by

$$\widetilde{M}(D(f)) = M[1/f]$$

with the evident action of $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R[1/f]$.

Definition 1.14. *Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is called quasi-coherent if for every point $x \in X$ there exist an affine neighborhood $U \cong \mathrm{Spec}(R)$ such that $\mathcal{M}|_U$ is as an $\mathcal{O}_U = \mathcal{O}_{\mathrm{Spec}(R)}$-module equivalent to $\tilde{M}$ for some R-module M. The category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X is defined as the full-subcategory of $\mathcal{O}_X$-module scheaves consisting of the quasi-coherent sheaves.*

Theorem 1.15. *For every ring R the functor*

$$\mathrm{Mod}_R \to \mathrm{QCoh}(\mathrm{Spec}(R)) \qquad M \mapsto \widetilde{M}$$

from R-modules to quasi-coherent sheaves is an equivalence of categories with inverse given by taking global sections.

Finally one important aspect of (quasi-coherent) modules is that they also satisfy descent. Here descent is, similar to the descent that sheaves satisfy, a local-to-global principle.

Let us describe this for modules: assume that we have a ring R and an open cover $U_i = \mathrm{Spec}(R_i)$ with double intersections $U_{ij} = \mathrm{Spec}(R_{ij})$. Here for simplicity we assume that double intersections are also affine, which is for example true of all the open sets are principal.

Then the descent result is that an R-module M is essentially the same as the following data:

(1) A family of R_i-modules M_i.
(2) For every pair i, j an isomorphism

$$\varphi_{ij} : M_i \otimes_{R_i} R_{ij} \xrightarrow{\sim} M_j \otimes_{R_j} R_{ij}$$

such that after basechange to R_{ijk} we have $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ for all triples i, j, k (this in particular implies $\varphi_{ii} = \mathrm{id}$ which is sometimes stated explicitly). This latter condition is called the cocycle condition or cocycle identity.

More precisely this correspondence is given by sending an R-module M to the family $M_i := M \otimes_R R_i$ together with the isomorphisms

$$\begin{aligned} M_i \otimes_{R_i} R_{ij} &\cong (M \otimes_R R_i) \otimes_{R_i} R_{ij} \\ &\cong M \otimes_R R_{ij} \\ &\cong (M \otimes_R R_i) \otimes_{R_i} R_{ij} \\ &\cong M_j \otimes_{R_j} R_{ij} \ . \end{aligned}$$

So we claim that evert family of modules satisfying (1) and (2) above arises from a family as below. A similar statement holds true for morphisms between modules. This is very useful since it allows us to define and manipulate modules locally.

The proof of this descent statement for modules proceeds by proving a similar descent statement for sheaves and then deducing the module statement from that using that modules are equivalent to quasi-coherent sheaves. So the more intuitive and geometric statement is the one that sheaves also satisfy descent.

CONSTRUCTION 1.16. For a scheme X There is a tensor product $\otimes_{\mathcal{O}_X}$ on the category of $\mathcal{O}_X$-modules defining by the sheafiffcation of the 'pointwise' tensor prodduct. This tensor product restrict to a tensor product on the full subcategory of quasi-coherent sheaves. For $X = \mathrm{Spec}(R)$ it corresponds under the equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R$ to the tensor product of R-modules.

DEFINITION 1.17. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{L}$ is called invertible or a line bundle over X, if there exists an $\mathcal{O}_X$-module $\mathcal{L}'$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \cong \mathcal{O}_X$.*

We had stated the following results (without proof):

PROPOSITION 1.18.
1. *Every invertible $\mathcal{O}_X$-module is quasi-coherent.*
2. *If $X = \mathrm{Spec}(R)$ is affine then invertible $\mathcal{O}_X$-modules are via taking global sections equivalent to invertible R-modules.*
3. *Given an open cover of X. Then an $\mathcal{O}_X$-module $\mathcal{M}$ is invertible precisely if it restricts to invertible modules on all of the open sets.*
4. *For any line bundle $\mathcal{L}$ over X there exists an open cover such that the restriction to all the open sets are trivial line bundles.*

The first thing we want to do is to give proofs of these facts and understand line bundles better.

2. Line bundles and vector bundles

Our first goal is to prove and understand Proposition 1.18 a bit better. Therefore let us try to prove these results. The first thing we need to do discuss the notion of epimorphisms in categories

DEFINITION 2.1. *A morphism $\pi : M \to N$ in a category is called an epimorphism if for every pair of morphisms $f, g : N \to O$ with $f\pi = g\pi$ we already have that $f = g$.*

EXAMPLE 2.2.
1. If a morphism $\pi : M \to N$ admits a section, that is a morgphism $s : N \to M$ with $\pi s = \mathrm{id}_N$, then it is an epimorphism.
2. In the category of sets the epimorphisms are precisely the surjections. Clearly surjections are epimorphisms, since they admit sections. Conversely if $\pi : M \to N$ is not surjective, say $n \in N$ is not in the image, then we can find maps $f, g : N \to \{0, 1\}$ which send $N \setminus \{n\}$ to 0, and such that $f(n) = 0$ and $f(n) = 1$. Thus $f\pi = g\pi$ but $f \neq g$.
3. In the category of R-modules a morphism $\pi : M \to N$ is an epimorphism precisely if it is surjective. The direction that surjections are epimorphisms follows easily from the last example since equality of maps can be checked on underlying set valued maps. For the converse assume that $\pi : M \to N$ is not surjective. Then $\mathrm{Im}(\pi) \neq N$ and thus $N/\mathrm{Im}(\pi) \neq 0$. Thus we can consider the maps $N \to N/\mathrm{Im}(\pi)$ given by the projection and the zero maps. These agree after composition with M but are different.
4. In the category of rings epimorphisms are not necessarily surjective. For example $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism since morphisms $\mathbb{Q} \to R$ are given by morphisms $f : \mathbb{Z} \to R$ such that $f(n)$ is invertible for all $n \in \mathbb{Z} \setminus \{0\}$.

Proposition 2.3. *A morphisms $\pi : M \to N$ is an epimorphism precisely of the square*

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\pi"'] & N \arrow[d, "\mathrm{id}"] \\ N \arrow[r, "\mathrm{id}"] & N \end{tikzcd}$$

is a pushout.

Proof. Assume the square is a pushout and we are given $f, g : N \to O$ such that $\pi f = \pi g$. Then by the universal property of the pushout we get a map $h : N \to O$ such that $f = h = g$.
Conversely assume that π is an epi. Then the universal property follows immediately. □

Corollary 2.4. *Every left adjoint functor preserves epimorphism, in particular equivalences of categories preserve and detect epimorphisms.*

Corollary 2.5. *A morphism of sheaves of rings/sets/modules $F \to G$ is an epimorphism preicsely if it induces epimorphisms on stalks*

Proof. We have to verify that the square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \\ G \arrow[r] & G \end{tikzcd}$$

is a pushout. But this is the case precisely if it is a pushout on stalks (since stalks preserve colimits and detect equivalences), thus precisely if the induced morphisms on stalks are epimorphisms. □

Warning 2.6. *An epimorphism of sheaves $F \to G$ need not be surjective on sections $F(U) \to G(U)$. For example for any topological space X we have the morhphism $\exp : C^0(-, \mathbb{C}) \to C^0(-, \mathbb{C}^\times)$ given by composition with the exponential function. This is an epimorphism of sheaves since locally the logarithm exists. But globally it doesn't for example for $X = \mathbb{C}^\times$. The identity $\mathbb{C}^\times \to \mathbb{C}^\times$ does not lift through $\exp : \mathbb{C} \to \mathbb{C}^\times$.*

Definition 2.7. *Let X be a scheme and $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{M}$ is of finite type if for every point x there exists an open neighborhood and an epimorphism $\mathcal{O}_U^n \to \mathcal{M}|_U$ of $\mathcal{O}_U$-modules.*

Let us unfold what that means. First observe that for a scheme X, a $\mathcal{O}_X$-module map $\mathcal{O}_X \to \mathcal{M}$ is the same as a global section of $\mathcal{M}$: Indeed, a $\mathcal{O}_X$-module map $\mathcal{O}_X \to \mathcal{M}$ assigns to every open U a $\mathcal{O}_X(U)$-module map $\mathcal{O}_X(U) \to \mathcal{M}(U)$ (compatibly), i.e. an element of $\mathcal{M}(U)$. But by the sheaf property, a compatible choice of elements $\mathcal{M}(U)$ for each open U is just determined by a global section. Similarly, a map $\mathcal{O}_X^n \to \mathcal{M}$ is the same as an n-tuple of global sections.
An epimorphism $\mathcal{O}_X^n \to \mathcal{M}$ is therefore the same as n sections $s_1, ..., s_n \in \mathcal{M}(X)$, such that for each point $x \in X$ the germs of the sections $(s_1)_x, .., (s_n)_x$ generate $\mathcal{M}_x$. Thus, $\mathcal{M}$ is of finite type precisely if for each x, we find a neighbourhood U and sections $s_1, \ldots, s_n \in \mathcal{M}(U)$ such that $s_1, \ldots, s_n$ generate the stalks $\mathcal{M}_{x'}$ for each $x' \in U$.

PROPOSITION 2.8.

(1) *Assume that $X = \mathrm{Spec}(R)$ and that $\mathcal{M}$ is quasi-coherent. Then $\mathcal{M}$ is of finite type precisely if the global sections $\mathcal{M}(X)$ are finitely generated as an R-module.*
(2) *Assume that $\mathcal{M}$ is quasi-coherent. Then $\mathcal{M}$ is of finite type precisely if for every $x \in X$ there exists an open affine neighborhood U such that $\mathcal{M}(U)$ is finitely generated as an $\mathcal{O}_X(U)$-module.*

PROOF. We first claim that a morphism between quasi-coherent sheaves $\tilde{M} \to \tilde{N}$ on $X = \mathrm{Spec}(R)$ is an epimorphism of $\mathcal{O}_X$-modules precisely if the map $M \to N$ is surjective. To see this we simply observe that being an epimorphism of $\mathcal{O}_X$-modules is a priori a stronger condition as we have to test against more morphisms (in general if one has an epimorphism in a category then it is also an epimorphism in each full subcategory in which it lies). Thus if it is an epimorphism of $\mathcal{O}_X$-modules then it clearly is an epimorphism of modules, hence surjective. The converse follows by observing that being surjective on global sections certainly implies being surjective on stalks (as they are localisations).
So $\mathcal{M}(U)$ is finitely generated as $\mathcal{O}_X(U)$-module if and only if we find an epimorphism $\mathcal{O}_U^n \to \mathcal{M}|_U$. Thus (2) is directly equivalent to the definition of finite type.
We also deduce one direction of (1), namely that if global sections $\mathcal{M}(X)$ are finitely generated that $\mathcal{M}$ is finite type. Thus it remains to show the other direction of (1), namely that if $\mathcal{M}$ has the property that on an open cover the sections $\mathcal{M}(U_i)$ are all finitely generated, that then also $\mathcal{M}(\mathrm{Spec}(R))$ is finitely generated. To this end we choose a finite affine cover $U_i = \mathrm{Spec}(R[1/f_i])$ and local sections $s_{ij} \in M[1/f_i]$ that generate. We can without loss of generality assume that s_{ij} is in R since we can always multiply by f_i without destroying the property of generating. Then we claim that the collection of all $s_{ij} \in M$ generate M. This follows since they generate locally, for example by translating back to a map $\mathcal{O}_X^{\sum n_i} \to \mathcal{M}$ of quasicoherent sheaves, which is surjective on stalks by construction. $\square$

PROPOSITION 2.9. *Every line bundle $\mathcal{L}$ on a scheme X is of finite type.*

PROOF. We choose a line bundle $\mathcal{L}^{-1}$ which is inverse to $\mathcal{L}$ and an isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to \mathcal{O}_X$. Then because of the definition of the tensor product as a sheafification we locally around each point find finitely many sections $s_i \in \mathcal{L}(U)$ and $t_i \in \mathcal{L}^{-1}(U)$ such that the element

$$\sum s_i \otimes t_i \in (\mathcal{L} \otimes \mathcal{L}^{-1})(U)$$

maps to the unit element 1 under this isomorphism (note that this is not true globally by the fact that we have to sheafify the tensor product but locally since sheafification doesn't change stalks).
Now we consider the morphism

$$s : \mathcal{O}_U^n \to \mathcal{L}|_U$$

given by the sections $s_1, ..., s_n$. After tensoring with $\mathcal{L}^{-1}$ this induces a morphism

$$\mathcal{O}_U^n \otimes \mathcal{L}^{-1} \to \mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_U$$

which is an epimorphism since 1 is in the image. Then it follows that the initial morphism s, which can be recovered by tensoring with $\mathcal{L}$, was already an epimorphism. (For example since we can detect epimorphisms on stalks, and on stalks the tensor product is really the tensor product, and tensor products preserve surjective maps) □

LEMMA 2.10. *Assume that for any $\mathcal{O}_X$-module sheaf $\mathcal{M}$ we have an epimorphism $\mathcal{M} \to \mathcal{O}_X$. Then it is locally split, meaning that every point has a neighbourhood U such that we find a map $\mathcal{O}_U \to \mathcal{M}|_U$ such that the composite $\mathcal{O}_U \to \mathcal{M}|_U \to \mathcal{O}_U$ is the identity.*

PROOF. Since it is an epimorphism we locally find a preimage of 1, that it around each point x we find an open U with $s \in \mathcal{M}(U)$ such that $s \mapsto 1 \in \mathcal{O}_X(U)$. But this then already constitutes a map as required. □

PROPOSITION 2.11. *Every line bundle $\mathcal{L}$ over X is locally a summand of $\mathcal{O}_X^n$ for some n (by this we mean that there for every $x \in X$ an open neighborhood U such that $\mathcal{L}|U$ is a summand of $\mathcal{O}_U^n = \mathcal{O}_X^n|_U$). In particular it is quasi-coherent.*

PROOF. By the fact that the line bundle is of finite type we find locally on X an epimorphism $\mathcal{O}_X^n \to \mathcal{L}$. After tensoring with $\mathcal{L}^{-1}$ this epimorphism is by the previous lemma locally split, so after restriction to a further open subset we can assume that it is split. But then we deduce that the initial morphism (which can be written by further tensoring with $\mathcal{L}$) is also split. But this then shows that $\mathcal{L}$ is a retract of the quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X^n$. But this now implies that $\mathcal{L}$ is quasi-coherent by Exercise 1, exercise sheet 1. □

COROLLARY 2.12. *For an affine scheme $X = \operatorname{Spec}(R)$ line bundles over $\operatorname{Spec}(R)$ are via global sections the same as invertible R-modules.*

DEFINITION 2.13. *A vector bundle on a scheme is a locally free $\mathcal{O}_X$-module.*

Note that every vector bundle is quasi-coherent and of finite type.

THEOREM 2.14. *Every line bundle is a vector bundle, in fact line bundles are locally free of rank 1, that is locally isomorphic to $\mathcal{O}_X$. Conversely, a vector bundle locally free of rank 1 is already a line bundle.*

PROOF. For the first statement, it suffices to work locally. Thus assume $X = \operatorname{Spec}(R)$ is affine and the line bundle $\mathcal{L}$ corresponds to an invertible R-module L. Thus $L \otimes_R L^{-1} \cong R$. For any point $x \in \operatorname{Spec}(R)$, we want to find a neighbourhood of x in which L is free. Observe that $L \otimes_R \kappa(x)$ and $L^{-1} \otimes_R \kappa(x)$ are inverse $\kappa(x)$-modules, thus 1-dimensional. We can lift a basis element to an element $a \in L \otimes_R R[1/f]$. By the prime ideal version of the Nakayama Lemma (last semester, Corollary 16.9), there exists a smaller neighbourhood in which a actually generates L, i.e. after changing f, we have that a generates $L \otimes_R R[1/f]$.
We now consider this as a map $\mathcal{O}_X|_{D(f)} \to \mathcal{L}|_{D(f)}$, which is an epimorphism. By Lemma 2.10 (as used in the proof of Proposition 2.11), we can (in a possibly smaller neighbourhood) find a split

$$\mathcal{L}|_{D(f)} \to \mathcal{O}_X|_{D(f)}. \tag{10}$$

Since the composite $\mathcal{L}|_{D(f)} \to \mathcal{O}_X|_{D(f)} \to \mathcal{L}|_{D(f)}$ is the identity, the map (10) is an isomorphism at x, thus again by Nakayama, it is surjective on sections in a yet smaller neighbourhood. It is also injective (being a retract), thus bijective on sections in some neighbourhood.
For the converse, assume $\mathcal{L}$ is a vector bundle locally of rank 1. We define a dual sheaf $\mathcal{L}^\vee$ by letting $\mathcal{L}^\vee(U)$ be the set of $\mathcal{O}_U$-module sheaf morphisms $\mathcal{L}|_U \to \mathcal{O}_U$.

Note that $\mathcal{L}^\vee$ is a $\mathcal{O}_X$-module sheaf, and it is quasicoherent, since $\mathcal{L}$ is locally free, and on an U_i where $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$, we get $\mathcal{L}^\vee|_{U_i} \cong \mathcal{O}_{U_i}$. We have a canonical map

$$\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$$

which is locally an isomorphism. It follows that $\mathcal{L}$ is a line bundle. □

We now are ready to state and prove the following result that was already sketched last term.

THEOREM 2.15. *For any scheme X we have a natural isomorphism*

$$\mathbb{P}^n(X) := \mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{P}^n) \cong \{\mathcal{L} \subseteq \mathcal{O}_X^{n+1} \mid \mathcal{L} \textit{ invertible}, \ \textit{locally complementable}\}$$

Here the inclusion means that $\mathcal{L}$ is a subsheaf of $\mathcal{O}_X^{n+1}$ as an $\mathcal{O}_X$-module sheaf, that is for each U we have that $\mathcal{L}(U) \subseteq \mathcal{O}_X^{n+1}(U)$ and that this subset is closed under restriction.

PROOF. Recall that in Theorem 15.12, we produced a bijection

$$\mathbb{P}^n(\mathrm{Spec}(R)) \cong \{\mathcal{L} \subseteq \mathcal{O}_{\mathrm{Spec}(R)}^{n+1} \mid \mathcal{L} \text{ invertible}, \ \text{locally complementable}\},$$

which by construction was natural in R. Thus, we have already established this for *affine* schemes. We generalize to all schemes *by descent*: For fixed X, we now consider

$$\mathcal{F} : (U \subseteq X) \mapsto \mathrm{Hom}_{\mathrm{Sch}}(U, \mathbb{P}^n)$$

and

$$\mathcal{G} : (U \subseteq X) \mapsto \{\mathcal{L} \subseteq \mathcal{O}_U^{n+1} \mid \mathcal{L} \text{ invertible}, \ \text{locally complementable}\}$$

as set-valued presheaves on X. They are in fact sheaves: For $\mathcal{F}$ this is clear, for $\mathcal{G}$ this follows from the fact that we can check being invertible and locally complementable locally. (The first by the identification of line bundles with rank 1 vector bundles, the second by definition).

Now we get a unique morphism $\varphi : \mathcal{F} \to \mathcal{G}$ which on *affine* $U \subseteq X$ is given by the bijection $\mathcal{F}(U) \to \mathcal{G}(U)$ from Theorem 15.12. This follows from the following observation: For an arbitrary open U and $a \in \mathcal{F}(U)$, there exists a unique element $b \in \mathcal{G}(U)$ with restrictions $b|_{U_i} = \varphi(a|_{U_i})$ to all affine opens $U_i \subseteq U$. By the sheaf condition, there exists at most one such element. To see that one exists, we need to show that for two different affines U_i, U_j with intersection U_{ij},

$$\varphi(a|_{U_i})|_{U_{ij}} = \varphi(a|_{U_j})|_{U_{ij}}.$$

in $\mathcal{G}(U_{ij})$. Now U_{ij} is *not* necessarily affine, but can be covered by affines V_k. The restrictions $\varphi(a|_{U_i})|_{V_k}$ and $\varphi(a|_{U_j})|_{V_k}$ both agree with $\varphi(a|_{V_k})$ by naturality of φ on affines, and so the claim follows.

Now φ is a morphism of sheaves on X, but it is an isomorphism on affine opens, so it is an isomorphism of sheaves, in particular on global sections. □

Theorem 2.15 allows us to think of $\mathbb{P}^n$ as a kind of *classifying space*: Every map $X \to \mathbb{P}^n$ corresponds to a line bundle on X (with locally complementable embedding to $\mathcal{O}_X^{n+1}$). In particular, the identity $\mathrm{id}_{\mathbb{P}^n} : \mathbb{P}^n \to \mathbb{P}^n$ gives rise to a *tautological* line bundle on $\mathbb{P}^n$, which we denote by $\mathcal{O}(-1)$ with given locally complementable embedding $\mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1}$. By naturality of the bijection from Theorem 2.15, the bijection is also given by

$$(f : X \to \mathbb{P}^n) \mapsto (f^*\mathcal{O}(-1) \to f^*(\mathcal{O}_{\mathbb{P}^n}^{n+1}) = \mathcal{O}_X^{n+1})$$

Definition 2.16. *We define line bundles $\mathcal{O}(d)$ on $\mathbb{P}^n$ by:*

$$\mathcal{O}(d) = \mathcal{O}(-1)^{\otimes(-d)}.$$

In order to describe them more explicitly, recall that $\mathbb{P}^n$ is covered by affine opens U_i for $i = 0, \dots, n$, with

$$U_i = \mathrm{Spec}(\mathbb{Z}[X_{0,i}, \dots, X_{n,i}]/(X_{i,i} - 1)) \cong \mathbb{A}^n$$

and $U_{i,j} = D(X_{j,i}) \subseteq U_i$, with gluing isomorphisms $\varphi_{i,j} : U_{j,i} \cong U_{i,j}$ induced by $X_{s,i} \mapsto X_{s,j} \cdot X_{i,j}^{-1}$.
A line bundle on $\mathbb{P}^n$ is thus described by line bundles $\mathcal{L}_i$ on all U_i, and isomorphisms $\psi_{i,j} : \mathcal{L}_j|_{U_{j,i}} \to \varphi_{i,j}^* \mathcal{L}_i|_{U_{i,j}}$ satisfying a cocycle condition.

Proposition 2.17. *The line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$ admits a cocycle description as follows:*

- *On each open U_i we have a trivial line bundle.*
- *The gluing isomorphism $\psi_{i,j}$ is given (on sections) by multiplication with the unit $X_{i,j}^{-d}$.*

Proof. For $\mathrm{Spec}(R) \to \mathbb{P}^n$, the line bundle we assigned to this map (i.e. the pullback of $\mathcal{O}(-1)$) was given on an affine open $\mathrm{Spec}(R_i) \subseteq \mathrm{Spec}(R)$ which gets mapped into $U_i = \mathrm{Spec}(\mathbb{Z}[X_{0,i}, \dots, X_{n,i}]/(X_{i,i} - 1))$ as the submodule of R_i^{n+1} generated by the image of $(X_{0,i}, \dots, X_{n,i})$.
So $\mathcal{O}(-1)(U_i)$ gets canonically identified with the submodule of $\mathcal{O}(U_i)^{n+1}$ generated by $(X_{0,i}, \dots, X_{n,i})$. In particular, $\mathcal{O}(-1)$ is free on the U_i. Furthermore, observe that $\mathcal{O}(-1)(U_{j,i})$ is the submodule of $\mathcal{O}(U_{j,i})^{n+1}$ generated by $(X_{0,j}, \dots, X_{n,j})$, and $(\phi_{i,j}^* \mathcal{O}(-1))(U_{j,i})$ is the submodule generated by

$$(\phi_{i,j}(X_{0,i}), \dots, \phi_{i,j}(X_{n,i})) = X_{i,j}^{-1} \cdot (X_{0,j}, \dots, X_{n,j}).$$

So $\psi_{i,j}$ takes

$$a \cdot (X_{0,j}, \dots, X_{n,j}) \mapsto (a \cdot X_{i,j}) \cdot (\phi_{i,j}(X_{0,i}), \dots, \phi_{i,j}(X_{n,i})).$$

This shows that $\mathcal{O}(-1)$ has the desired gluing isomorphism, on $\mathcal{O}(n) = \mathcal{O}(-1)^{\otimes -n}$ we correspondingly get multiplication by $X_{i,j}^{-n}$. ☐

Example 2.18. A global section of $\mathcal{O}(d)$ is given by compatible global sections on each U_i, i.e.:

- An element a_i of $\mathbb{Z}[X_{0,i}, \dots, X_{n,i}]/(X_{i,i} - 1)$ for each i,
- such that $X_{i,j}^{-d} a_j$ and $\varphi_{i,j} a_i$ agree in $\mathbb{Z}[X_{0,j}, \dots, X_{i,j}^{\pm 1}, \dots, X_{n,j}]/(X_{j,j} - 1)$.

For $d = 1$ and fixed k, $a_i = X_{k,i}$ defines such a section. We write this section as $x_k \in \mathcal{O}(1)(\mathbb{P}^n)$.

Definition 2.19. *For a scheme X with line bundle $\mathcal{L}$ and global section $f \in \mathcal{L}(X)$, we let $D(f) \subseteq X$ of all $x \in X$ such that f is nonzero in $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.*

Note that for $X = \mathrm{Spec}(R)$ and $\mathcal{L} = \mathcal{O}_X$, $\mathcal{L}(X) = R$, and this definition agrees with the original definition of $D(f)$.

Lemma 2.20.

(1) *$D(f)$ is open.*
(2) *$\mathcal{L}$ is trivial on $D(f)$.*

(3) *There exists a section* $f^{-1} \in \mathcal{L}^{-1}(D(f))$ *such that the isomorphism*
$$\mathcal{L}^{-1} \otimes \mathcal{L} \to \mathcal{O}_X$$
takes $f^{-1} \otimes f \mapsto 1$ *on* $D(f)$.

PROOF. The first statement can be checked locally, so we assume that $X = \mathrm{Spec}(R)$ is affine and that $\mathcal{L}$ is the trivial line bundle $\mathcal{O}_X$. Then $D(f)$ agrees with the principal open $D(f) \subseteq \mathrm{Spec}(R)$, thus is open. For the second statement observe that the map
$$f : \mathcal{O}_X|_{D(f)} \to \mathcal{L}|_{D(f)}$$
is on stalks given by a map $\mathcal{O}_{X,x} \to \mathcal{L}_x$ which is an isomorphism after tensoring with $\kappa(x)$. Since $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules, we can noncanonically identify this map with multiplication with an element of $\mathcal{O}_{X,x}$ which is nonzero in $\kappa(x)$. This means it is a unit, so f induces isomorphisms on stalks, and thus is a sheaf isomorphism. The inverse map
$$\mathcal{L}|_{D(f)} \to \mathcal{O}_X|_{D(f)}$$
can, after tensoring with $\mathcal{L}^{-1}$, be identified with a section of $\mathcal{L}^{-1}$ that has the desired property. □

Note that we can generally multiply sections of line bundles: $f_1 \in \mathcal{L}_1(X)$ and $f_2 \in \mathcal{L}_2(X)$ multiply to an element $(f_1 \cdot f_2) \in (\mathcal{L}_1 \otimes \mathcal{L}_2)(X)$. For an $f \in \mathcal{L}(X)$, we think of $D(f)$ as the locus where f is invertible, in the sense that there exists an inverse section f^{-1} of $\mathcal{L}^{-1}$.

PROPOSITION 2.21. *On* $\mathbb{P}^n$, *we have:*

(1) *The standard opens* U_i *are given by* $D(x_i)$, *where* $x_i \in \mathcal{O}(1)(\mathbb{P}^n)$.
(2) *On* $D(x_i)$, *we have* $x_j \cdot x_i^{-1} = X_{j,i}$. *Here we interpret* $x_j \cdot x_i^{-1}$ *as section of* $\mathcal{O}(1) \otimes \mathcal{O}(-1) \cong \mathcal{O}_{\mathbb{P}^n}$.

PROOF. By construction, x_k is given on U_i by the element $X_{k,i}$ (relative to the trivialisation of $\mathcal{O}(1)$ described in Proposition 2.17). So $D(x_k) \cap U_i = U_{i,k} = U_k \cap U_i$, and therefore $D(x_k) = U_k$.
On $D(x_i)$, x_j trivializes to $X_{j,i}$, and x_i by $X_{i,i} = 1$, so the second statement follows. □

The rank of a vector bundle $\mathcal{V}$ on a scheme X is a function
$$\mathrm{rk}_{\mathcal{V}} : X \to \mathbb{N}$$
of sets, which is given by sending each point $x \in X$ to the dimension of the stalk $\mathcal{V}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ as a $\kappa(x)$-vector space. In a neighboordhood of x we find a trivialization $\mathcal{V}|_U = \mathcal{O}_X^n|_U$. This shows that the rank is everywhere on U the same (and finite), i.e. that the rank is a locally constant function. Of course it does not have to be globally constant. [1] Note that for a line bundle the function is the constant function 1.
We finish this section by a characterisation of vector bundles on affine schemes.

DEFINITION 2.22. *An object* P *in a category is called projective if every epimorphism* $M \to P$ *admits a section.*

EXAMPLE 2.23.

[1] Of course one could define a rank-function for every $\mathcal{O}_X$-module sheaf of finite type, but it would generally not be locally constant only upper semicontinuous.

(1) Every set is projective in the category of sets (we assume the axiom of choice).
(2) Every K-vector space is projective. To see this we observe that it is enough to define a section of $M \to V$ on a basis of V where we can simply pick preimages.
(3) More generally: Free modules over a ring are projective.
(4) There are also non-free examples: For example, over a ring $R \times S$, R as $R \times S$-module is always projective.
(5) The $\mathbb{Z}$-module $\mathbb{Z}/2$ is not projective, since the surjection $\mathbb{Z} \to \mathbb{Z}/2$ does not admit a section.

Lemma 2.24. *An R-module M is (finitely generated) projective if and only if it is a retract of a (finitely generated) free R-module.*

Proof. Assume P is projective. Choose a surjection $R^I \to P$ from a free R-module (by choosing a set of generators). This map admits a section, so P is a retract of R^I.
Conversely, assume P is a retract of R^I, i.e. we have maps $i : P \to R^I$ and $r : R^I \to P$ with composite $r \circ i = \mathrm{id}_P$. Given a surjection $M \to P$, choose a lift

$$\begin{tikzcd} & M \arrow[d] \\ R^I \arrow[ur, "f"] \arrow[r, "r"] & P \end{tikzcd}$$

(we can do this since R^I is free.) Then $f \circ i$ is the desired section of $M \to P$. $\square$

Theorem 2.25. *A quasi-coherent sheaf $\mathcal{M}$ over an affine scheme* $\mathrm{Spec}(R)$ *is a vector bundle precisely if* $\mathcal{M}(\mathrm{Spec}(R))$ *is a finitely generated projective module over R. In particular projective modules are locally free.*

Proof. If M is a finitely generated projective R-module, then $M \otimes_R S$ is a finitely generated projective S-module for each ring homomorphism $R \to S$. Let $x \in \mathrm{Spec}(R)$ be a point, and let $a_1, \dots, a_n$ be a basis of $M \otimes_R \kappa(x)$. We can lift $a_1, \dots, a_d$ to a neighbourhood, and by the prime ideal version of Nakayama (Corollary 16.9), we find a neighbourhood $M \otimes_R R[1/f]$ where $a_1, \dots, a_d$ are generators. We thus have a surjective map $r : R[1/f]^n \to M \otimes_R R[1/f]$, which admits a section i by projectivity. After tensoring with κ, $r \circ i = \mathrm{id}$, so by another application of Nakayama, i is surjective in a smaller neighbourhood. Thus we can choose f so that $R[1/f]^n \to M \otimes_R R[1/f]$ is an isomorphism, so $\widetilde{M}$ is a vector bundle.
For the other direction, let $\widetilde{M}$ be a vector bundle on $\mathrm{Spec}(R)$. By Proposition 2.8, M is finitely generated. Choose a surjective map $R^n \to M$, i.e. an epimorphism $\mathcal{O}_X^n \to \widetilde{M}$. We would like to construct a section $s : M \to R^n$. We define a sheaf

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{O}_X^n)(U) = \{\mathcal{O}_U \text{ module homomorphisms } \widetilde{M}|_U \to \mathcal{O}_X^n|_U\},$$

and $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{M})$ analogously. Since $\widetilde{M}$ is locally a free $\mathcal{O}_U$-module, we see that these are vector bundles, in particular quasicoherent. The map

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{O}_X^n) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{M})$$

arising from postcomposition with the morphism $\mathcal{O}_X^n \to \widetilde{M}$ is surjective on stalks, therefore an epimorphism. Since global sections give

$$\mathrm{Hom}_R(M, R^n) \to \mathrm{Hom}_R(M, M),$$

which is therefore surjective, we thus see that the map $R^n \to M$ admits a section. Thus M is projective. $\square$

3. Closed immersions

Definition 3.1. *A map $f : Z \to X$ of schemes is called a closed immersion if the induced map on topological spaces is a closed immersion (a homeomophism onto a closed subset) and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Z$ is an epimorphism of sheaves.*

Proposition 3.2. *Let $f : Z \to X$ be a map of schemes, then TFAE:*

(1) *f is a closed immersion*
(2) *For all open subsets $U \subseteq X$ with $U = \operatorname{Spec}(A)$ we have that $f^{-1}(U) = \operatorname{Spec}(B) \subseteq Z$ is open affine and $A \to B$ is surjective.*
(3) *There exists and open cover of X by affine schemes which satisfy the property of part 2.*

Note that this in particular means that closed immersions $Z \to \operatorname{Spec}(A)$ are given by maps $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$ (and are in particular affine). We will prove the statement at the end of the section with the aid of some more definitions and results.

Definition 3.3. *Let $(f, f^\sharp)$ be a map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.*

(1) *If N is an $\mathcal{O}_X$-module, then the pushforward $f_*(N)$ is the $\mathcal{O}_Y$-module with the structure morphism*
$$\mathcal{O}_Y \times f_*(N) \to f_*(\mathcal{O}_X) \times f_*(N) = f_*(\mathcal{O}_X \times N) \to f_*(N)\ .$$
This is clearly a $\mathcal{O}_Y$-module.
(2) *If M is an $\mathcal{O}_Y$-module, then $f^{-1}M$ is a sheaf of $f^{-1}\mathcal{O}_X$-modules via the map*
$$f^{-1}(\mathcal{O}_Y) \times f^{-1}(M) = f^{-1}(\mathcal{O}_X \times M) \to f^{-1}(M)\ .$$
We now define the pullback f^*M as the $\mathcal{O}_X$-module $f^*M = f^{-1}M \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Some people/books write f^* when they mean f^{-1} but we will only use f^* to mean the pullback of $\mathcal{O}_X$-modules we defined above.

Proposition 3.4. *The functor f^* is left adjoint to f_*. We have the following equivalences:*
$$f^*(\mathcal{O}_Y) \cong \mathcal{O}_X$$
$$f^*(M \otimes_{\mathcal{O}_Y} N) \cong f^*(M) \otimes_{\mathcal{O}_X} f^*(N)$$

Proof. We recall that on the level of sheaves we have that the functor f^{-1} is left adjoint to f_*. As a result we have a natural isomorphism
$$\operatorname{Hom}_{\mathrm{Shv}(X)}(f^{-1}M, N) \cong \operatorname{Hom}_{\mathrm{Shv}(Y)}(M, f_*N)$$
Now the $\mathcal{O}_Y$-linear maps on the right correspond to $f^{-1}\mathcal{O}_X$-linear maps on the left, so that we get a natural isomorphism
$$\operatorname{Hom}_{\mathrm{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}M, N) \cong \operatorname{Hom}_{\mathrm{Mod}(\mathcal{O}_Y)}(M, f_*N)$$
This can already be interpreted as saying that the adjunction statement is true for the morphism of locally ringed spaces $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ for which no further

tensoring is needed. Now we simply note that we also have by the usual adjunction argument a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}M, N) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(f^{-1}M \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, N)$$

which gives the desired statement.
The first of the three isomorphism follows immediately by definition:

$$f^*(\mathcal{O}_Y) = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X \ .$$

For the second we have

$$\begin{aligned} f^*(M \otimes_{\mathcal{O}_Y} N) &= f^{-1}(M \otimes_{\mathcal{O}_Y} N) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &= (f^{-1}(M) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}N) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &= f^*(M) \otimes_{\mathcal{O}_X} f^*(N) \end{aligned}$$

where for the second isomorphism we have used that f^{-1} commutes with tensor products and colimits (which is immediate to check and omitted here).

$\square$

PROPOSITION 3.5. *The pullback $f^*(M)$ of a quasi-coherent sheaf M is quasi-coherent. If M is a vector bundle (line bundle) then it is a vector bundle (line bundle). If $f : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ then the induced functor $f^* : \mathrm{Mod}_S \to \mathrm{Mod}_R$ is given by the base change $- \otimes_S R$.*

PROOF. For the first statement we note that the pullback functor commutes with restriction to an open subset. Thus we can check the statement locally and therefore assume without loss of generality that we are in the situtation of the second statement, i.e. $f : X = \mathrm{Spec}(R) \to \mathrm{Spec}(S) = Y$ and the sheaf is of the form $\mathcal{M} = \widetilde{M}$ for some S-module M. Then we have for any $\mathcal{O}_X$-module $\mathcal{N}$ natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(f^*(\widetilde{M}), \mathcal{N}) &= \mathrm{Hom}_{\mathcal{O}_Y}(\widetilde{M}, f_*(\mathcal{N})) \\ &= \mathrm{Hom}_S(M, f_*(\mathcal{N})(Y)) \\ &= \mathrm{Hom}_S(M, \mathcal{N}(X)) \\ &= \mathrm{Hom}_R(M \otimes_S R, \mathcal{N}(X)) \\ &= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M \otimes_S R}.\mathcal{N}) \ . \end{aligned}$$

Thus the Yoneda Lemma shows that $f^*(\widetilde{M}) \cong \widetilde{M \otimes_S R}$. We thereby have proven the first and the last statement. The statements about vector bundles and line bundles now either follow by using that base-change preserves projective and invertible modules or even by restriction to a trivializing open and noting that pullback preserves free modules (by the fact that it preserves sums and the one-dimensional trivial module). $\square$

Now in general the pushforward of quasi-coherent sheaves is not quasi-coherent (examples are somewhat pathological though and will not be discussed). Instead we need additional assumptions on the morphism. To this end we recall that a topological space is called quasi-compact if every open cover has a finite subcover.

DEFINITION 3.6. *A space X is called quasi-separated if given two quasi-compact open subsets U and V, then the intersection $U \cap V$ is also a quasi-compact open subset. A map of schemes is called quasi-compact (resp. quasi-separated) if the inverse images* *of quasi-compact (resp. quasi-separated) open subsets of X are quasi-compact (resp. quasi-separated).*

Example 3.7. Every affine scheme is quasi-compact and quasi-separated. To see this we only have to verify that it is quasi-separated. But quasi-compact open sets are finite unions of principal opens (since every open is a union of principal opens). Thus the intersection of quasi-compact opens is the finite union of intersections of principal opens, which are again quasi-separated.

We also note that open subsets of quasi-separated spaces are quasi-separated, thus all open subsets of affine schemes are quasi-separeted.

Example 3.8. Every morphism between affine schemes is quasi-compact and quasi-separated. To see this we note that pre-images of principal opens are principal open. Since compact opens are finite unions of those this shows that preimages of quasi-compact opens are again quasi-compact. For quasi-separatedness nothing is to show since all open subsets of affine schemes are quasi-separated.

Proposition 3.9. *Let $f : X \to Y$ be a map of schemes which is quasi-compact and quasi-separated, and let M be a quasi-coherent sheaf on Y , then $f_*(M)$ is a quasi-coherent sheaf on X. On affines the functor corresponds to the restriction of modules.*

Proof. let $U \subseteq Y$ be an affine open. We want to show that $f_*(M)|_U$ is quasi-coherent. This suffices since being quasi-coherent is a local property. By definition of f_* we have that

$$f_*(M)|_U = f_*(M|_{f^{-1}(U)})$$

where the second f means the restriction $f : f^{-1}(U) \to U$. Thus we can without loss of generality assume that $Y = \mathrm{Spec}(S)$ is affine. Since the morphism f is quasi-compact and quasi-separated it then follows that X is quasi-compact and quasi-separated, i.e.

$$X = \bigcup_{i=1}^{n} \mathrm{Spec}(R_i)$$

with

$$\mathrm{Spec}(R_i) \cap \mathrm{Spec}(R_j) = \bigcup_k \mathrm{Spec}(R_{ijk})$$

where the latter union is also finite. Now we set

$$N := f_*(M)(Y)$$

and want to show that the induced map

$$\widetilde{N} \to f_*(M)$$

is an isomorphism. It suffices to check that the map

$$N[1/s] = \widetilde{N}(D(s)) \to f_*(M)(D(s))$$

is an isomorphism for each $s \in S$. We note that by descent we have

$$N = f_*(M)(Y) = M(X) = \mathrm{eq}\left(\prod_i M(\mathrm{Spec}(R_i)) \rightrightarrows \prod_{i,j,k} M(\mathrm{Spec}(R_{ijk}) \right)$$

where all the products are finite. [2]
Now we use the fact that the localization functor $-[1/s]$ commutes with finite products and equalizers to deduce that

$$\begin{aligned} N[1/s] &= \mathrm{eq}\left(\prod_i M(\mathrm{Spec}(R_i))[1/s] \rightrightarrows \prod_{i,j,k} M(\mathrm{Spec}(R_{ijk}))[1/s]\right) \\ &= \mathrm{eq}\left(\prod_i M(\mathrm{Spec}(R_i) \cap f^{-1}D(s)) \rightrightarrows \prod_{i,j,k} M(\mathrm{Spec}(R_{ijk} \cap f^{-1}D(s)))\right) \\ &= M(f^{-1}D(s)) = f_*(M)(D(s)) \end{aligned}$$

where for the second equality we have used that M is quasi-coherent and for the one before the last that it is a sheaf.
For the statement about affines: one can either argue using that this is implied by the last proof (where we simply note that global sections in this case are easy to deteremine) or slightly more elegantly that the functor

$$f_* : \mathrm{Mod}_R = \mathrm{QCoh}(\mathrm{Spec}(R)) \to \mathrm{QCoh}(\mathrm{Spec}(S)) = \mathrm{Mod}_S$$

is right adjoint to the functor f^*. We already know that f^* is base-change of modules. Thus the right adjoint has to be given by restriction. □

Now we are finally ready to give the proof of Proposition 3.2. We recall that the statement was that for a morphism $f : Z \to X$ of schemes TFAE:

(1) f is a closed immersion
(2) For all open subsets $U \subseteq X$ with $U = \mathrm{Spec}(A)$ we have that $f^{-1}(U) = \mathrm{Spec}(B) \subseteq Z$ is open affine and $A \to B$ is surjective.
(3) There exists and open cover of X by affine schemes which satisfy the property of part 2.

Proof. Proof of Proposition 3.2 (1) $\Rightarrow$ (2): assume that f is a closed immersion. We first note that if $Z \to X$ is a closed immersion and $U \subseteq X$ is open, then also $Z \cap U \to Z \cap X$ is a closed immersion. This is immediate by the definition noting that the restriction of surjective morphisms of sheaves is again surjective. Thus in order to show (2) we can without loss of generality assume that $X = \mathrm{Spec}(A)$ is affine. Then $|Z| \subseteq |X|$ (the underlying topological spaces) is closed which implies that $|Z| = |\mathrm{Spec}(A/I)|$ and the map $Z \to X$ is quasi-compact and quasi-separated. We now deduce that

$$f_*(\mathcal{O}_Z)$$

is quasi-coherent, thus equal to $\widetilde{B}$ for the A-module $B = \mathcal{O}_Z(Z) =: B$. The map

$$\mathcal{O}_X \to f_*(\mathcal{O}_Z)$$

[2]Here we use a version of descent for the situation where we have a cover U_i a and refinements $U_{ijk} \subseteq U_i \cap U_j$. In this situation for every sheaf we have that

$$F(U) \to \mathrm{Eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j,k} F(U_{i,j,k})\right).$$

is an equalizer. This follows by noting that an element in the equalizer is given by (s_i) such that $s_i|_{U_{ijk}}$ agrees with $s_j|_{U_{ijk}}$ for each i, j, k. But then we use that U_{ijk} covers $U_i \cap U_j$ to conclude that then also $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and we have an element in the ordinary equalizer.

being surjective now implies (since both are quasi-coherent) that the map on global sections

$$A \to B$$

is surjective. We thus have to show that the canonical 'affinization' morphism $Z \to \mathrm{Spec}(B)$ is an isomorphism. By construction this is a morphism over X so that we have a canonical diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & \mathrm{Spec}(B) \arrow[dl] \\ & X & \end{tikzcd}$$

where the morphisms to X are closed immersions of topological space (for the right hand map this follows from the surjectivity of $A \to B$). Now assume we have a point $x \in \mathrm{Spec}(B) \setminus Z$. Then

$$f_*(\mathcal{O}_Z)_x = 0$$

since the stalk can be formed over open subsets disjoint from Z. At the same time we have that since $x \in \mathrm{Spec}(B)$ that

$$f_*(\mathcal{O}_Z)_x = (\widetilde{B}_x) \neq 0$$

(where the latter is generally true for localizations at prime ideals). We deduce that such an x cannot exist and we have that our map is a homeomorphism. It thus remains to show that $\mathcal{O}_Z = \mathcal{O}_{\mathrm{Spec}(B)}$. For an open $U = U' \cap \mathrm{Spec}(Z)$ we have

$$\mathcal{O}_Z(U) = f_*(\mathcal{O}_Z)(U') = (\widetilde{B})(U') = \mathcal{O}_{\mathrm{Spec}(B)}(U' \cap \mathrm{Spec}(B)) = \mathcal{O}_{\mathrm{Spec}(B)}(U) \ .$$

The implication (2) $\Rightarrow$ (3) is clear.
Finally for (3) $\Rightarrow$ (1) we note that since everything can be checked locally it suffices to check that surjections $A \to B$ of rings induced closed immersion $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. On underlying spaces this is clear. Thus we need to check that the map $\mathrm{Spec}(A) \to f_*(\mathcal{O}_{\mathrm{Spec}(B)})$ is surjective. Since both are quasi-coherent this follows from the statement on global sections. $\square$

We will also say that for a closed immersion $Z \to X$ that Z is a closed subscheme of X. We identify two such $Z \to X$ and $Z' \to X$ if they are isomorphic over X, i.e. if there is a commutative triangle

$$\begin{tikzcd} Z \arrow[rr, "\simeq"] \arrow[dr] & & Z' \arrow[dl] \\ & X & \end{tikzcd}$$

We can therefore always assume that the underlying space of Z is actually a subset of X by replacing $i : Z \to X$ by $Z' = (\mathrm{im}(i), i_*(\mathcal{O}_Z))$ which we will often do for convenience and switch between the perspectives freely. Then (equivlanece classes of) closed subschemes of $\mathrm{Spec}(A)$ are in bijection with ideals $I \subseteq A$.
In particular for a scheme $(X, \mathcal{O}_X)$ and a closed subset Z of X there is not a unique way to turn Z into a closed subscheme, i.e. define a structure sheaf $\mathcal{O}_Z$ on Z and a map $i^\sharp : \mathcal{O}_X \to i_*(\mathcal{O}_Z)$. For example, if $X = \mathrm{Spec}(A)$ then for two ideals I, I' the closed subset $\mathrm{Spec}(A/I)$ and $\mathrm{Spec}(A/I')$ agree precisely if I and I' have the same radical, but the subschemes are only the same if $I = I'$.

DEFINITION 3.10. *Let $(X, \mathcal{O}_X)$ be a scheme and $Z \subseteq X$ be a closed subset. A scheme structure on Z is given by a closed subscheme $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ whose underlying space is equal to Z.*

One can wonder if such a scheme structure even exists in general. We will prove that this is indeed the case.

PROPOSITION 3.11. *Let $(X, \mathcal{O}_X)$ be a scheme and $Z \subseteq X$ be a closed subset. Then*

(1) *Then there exists a minimal scheme structure on $(Z, \mathcal{O}_Z)$, that is an initial object in the category of scheme structures on Z.*
(2) *This scheme structure $(Z, \mathcal{O}_Z)$ is also reduced (as an abstract scheme) and it is the unique subscheme structure on Z that is reduced, we will denote it by $Z_{red} = (Z, \mathcal{O}_Z)$ and refer to it as the reduced subscheme structure.*
(3) *For a reduced scheme $(Y, \mathcal{O}_Z)$ a morphism of schemes $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is the same as a morphism of schemes $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that the morphism of underlying topological spaces factors through $Z \subseteq X$.*

PROOF. We will construct the sheaf $\mathcal{O}_Z$ locally. So assume that we are given an affine open $U \subseteq X$, then we want to construct $(\mathcal{O}_Z)|_{U \cap Z}$. To this end we simply write $U = \mathrm{Spec}(A)$ ant then $Z = V(I)$ for a unique redically closed ideal I. Then we define the sheaf $\mathcal{O}_Z|_{U \cap Z}$ as the structure sheaf of $\mathrm{Spec}(A/I) \subseteq \mathrm{Spec}(A)$. We claim that this defines by descent a well-defined sheaf on Z. To see this we have to verify that upon passage to a smaller affine open $\mathrm{Spec}(A') = U' \subseteq U = \mathrm{Spec}(A)$ the restriction of the sheaf $\mathcal{O}_{\mathrm{Spec}(A/I)}$ is given by the sheaf $\mathcal{O}_{\mathrm{Spec}(A'/I)}$. But this is clear since nilpotent elements can be detected locally.
In this way we have constructed a reduced scheme structure $(Z, \mathcal{O}_Z)$ on Z which was unique as such. Now we want to argue that it is initial. To this extend assume that we are given a second scheme structure $(Z, \mathcal{O}'_Z)$. Then we need to show that there is a unique way of extending the identity $Z \to Z$ to a morphism of schemes, i.e define a map $\mathrm{id}^\sharp : \mathcal{O}'_Z \to \mathcal{O}_Z$. But such a map can be constructed locally in which case we have to construct a map of schemes $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I')$ over $\mathrm{Spec}(A)$ for a given ideal I in A whose radical is I. But this is simply a map $A/I' \to A/I$ under A which exists uniquely since $I' \subseteq I$.
For the third statement we note that this can again be checkes locally in which case we end up studying maps

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A/I)$$

for a radically closed ideal $I \subseteq A$, or equivalently ring maps $A/I \to B$. But these are the same as maps $A \to B$ that take I to zero. The latter condition is for B radically closed equivalent the the assertion that the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ takes the closed set $\mathrm{Spec}(B)$ corresponding to a closed set contained in $V(I)$. $\square$

Note that if $Z = X$ then this recovers the reduced scheme structure on X discussed in Section 13.

4. Noetherian schemes

DEFINITION 4.1. *A scheme X is called noetherian, if it admits a finite open cover by affines $\mathrm{Spec}(A_i)$ where each A_i is a noetherian ring.*

PROPOSITION 4.2. *A noetherian scheme is quasi-compact and for each affine open $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is noetherian.*

Proof. Assume that $X = \bigcup \mathrm{Spec}(A_i) = U_i$ for A_i noetherian. As a finite union of quasi-compact spaces the space X is itself quasi-compact. Now for given affine open $U \subseteq X$ we have to verify that $A := \mathcal{O}_X(U)$ is noetherian. U is covered by finitely many U_i's by quasi-compactness. Moreover we can further cover each $U \cap U_i$ by finitely many opens U_{ik} which are principal in U_i, i.e. of the form $\mathrm{Spec}(A_i[1/f_k])$. Then as a localization of a noetherian ring, the ring $A_i[1/f_i] = A_{ik}$ is itsef notherian. As a result we have a finite cover $U_{i,k} = \mathrm{Spec}(A_{ik})$ of $U = \mathrm{Spec}(A)$ by spectra of noetherian rings and we want to show that A is noetherian. In other words: we can without loss of generality assume $X = \mathrm{Spec}(A)$ is affine and noetherian and have to show that A is then noetherian as a ring.
If $X = \mathrm{Spec}(A)$ is affine and $I \subseteq A$ is an ideal, then we consider I as an A-module and we want to show that it is finitely generated. We can view I as a quasi-coherent sheave on $\mathrm{Spec}(A)$. The restriction to $\mathrm{Spec}(A_i)$ is then the ideal in A_i generated by the image. Thus finitely generated and thus it follows from Proposition 2.8 that $\tilde{I}$ is of finite type, hence I finitely generated. □

Recall Definition 5.9: a topological space is called noetherian if every descending chain of closed subsets becomes constant. Also note that by Zorn's lemma this is equivalent to the assertion that every non-empty set of open subsets has a maximal element (cf. the proof of Proposition 2.3).

Lemma 4.3. *Assume a topological space T is noetherian. Then it is quasi-compact and quasi-separated. Moreover each subspace of a noetherian space is again noetherian.*

Proof. If $T = \bigcup_{i \in I} U_i$ for opens $U_i \subseteq T$. Consider the set consisting of the open sets

$$V_F = \bigcup_{i \in F} U_i \ .$$

for $F \subseteq I$ finite. This sequence has a maximal element V_{F_0} which then already has to cover T by maximality. Thus T is quasi-compact.
To see that any subset $T' \subseteq T$ is noetherian note that closed subsets $Z \subseteq T'$ are intersections $Z' \cap T'$. Thus for a given sequence

$$V_0 \supset V_1 \supset V_2 \supset ...$$

of closed subsets in T' we find subsets V_i' closed in T with $V_i' \cap T' = V_i$. But then we set

$$V_i'' := \bigcap_{j \leq i} V_j' \ .$$

These sets are again closed, nested and we still have $V_i'' \cap T' = V_i$. Therefore this seuqence becomes stationary, which shows that the initial sequence does.
As a consequence of what we have proven, we see that each subset of a noetherian space is quasi-compact. This immediately implies that the space is quasi-separated. □

Proposition 4.4. *The underlying topological space of a noetherian scheme is noetherian.*

Proof. We first note that $\mathrm{Spec}(A)$ for A a noetherian ring is a noetherian topological space. This follows directly from the fact that a given sequence of closed subsets

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq ...$$

immediately translates into a seuqence

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$$

of radically closed ideals. This then becomes eventually constant by the definition of being noetherian so that the initial sequence becomes constant as well.
Now we claim that if a topological space T has a cover by finitely many noetherian spaces T_i, then T is itself noetherian. To see this we consider a sequence $V_0 \supset V_1 \supset V_2 \supset ...$ and take the intersections

$$V_0 \cap T_i \supset V_1 \cap T_i \supset V_2 \cap T_i \supset ...$$

These became constant, say at stage n_i. Then the initial sequence becomes constant at stage $\max_i n_i$. □

In particular we see that for noetherian schemes we get a decomposition as in Proposition 5.13.

Remark 4.5. The converse of Proposition 4.4 does not hold: for a scheme X the underlying topological space can be noetherian without the scheme being noetherian. This can be exemplified in the affine case: if $\mathrm{Spec}(A)$ is a noetherian topological space, then every sequence

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$$

of radically closed ideals becomes eventually constant. But this does not imply that every sequence of ideals becomes eventually constant in general. For example consider the ring

$$R = \mathbb{Z}_2[\sqrt[n]{2} \mid n \in \mathbb{N}] \subseteq \overline{\mathbb{Q}_2} \ .$$

This ring is not noetherian as there are for examples chains of ideals

$$(2) \subsetneq (\sqrt[2]{2}) \subsetneq (\sqrt[4]{2}) \subsetneq ...$$

doesn't stabilize. However we have that $\mathrm{Spec}(R) = \varprojlim_n \mathrm{Spec}(\mathbb{Z}_2[\sqrt[n]{2}])$ has two points.

We think of being noetherian as a finiteness condition on schemes. Now we will discuss the corresponding finiteness condition for quasi-coherent sheaves. We will make the definition in the generality of ringed spaces (because we can, but we really only care about schemes...).

Definition 4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then a $\mathcal{O}_X$-module M is called coherent, if it is of finite type and for each open $U \subseteq X$ together with a morphism*

$$\mathcal{O}_U^n \to \mathcal{M}|_U$$

the kernel of this map is also of finite type. We denote the category of coherent sheaves by $\mathrm{Coh}(X) \subseteq \mathrm{Mod}(\mathcal{O}_X)$.

Warning 4.7. *In general the structure sheaf $\mathcal{O}_X$ is not coherent. Not even in the affine case!*

Lemma 4.8. *Every coherent sheaf on a scheme is quasi-coherent.*

Proof. Let $M \in \mathrm{Coh}(X)$. Then for every $x \in X$ we find an epi

$$\mathcal{O}_u^n \xrightarrow{\pi} M|_U$$

and after further shrinking U another epimorphism

$$\mathcal{O}_U^m \to \ker(\pi) \ .$$

Thus we find that in this neighborhood $M|_U$ is the cokernel of

$$\mathcal{O}_U^m \to \mathcal{O}_U^n$$

and therefore quasi-coherent. □

Proposition 4.9. *The image of a morphism between coherent sheaves is also coherent. The kernel and cokernel of coherent $\mathcal{O}_X$-modules is again coherent. Also any extension of coherent $\mathcal{O}_X$-modules is coherent.*[3]

Proof. Consider a morphism $\psi : M \to N$ of coherent sheaves. The image $M \to \mathrm{Im}(\psi) \to N$ receives an epimorphism from M and is thus of finite type. It is also a subsheaf of N and thus satisfies the second condition in the condition of being coherent. As an addendum we note that we have only used that M is of finite type.

Now we want to show that the kernel $\ker(\psi)$ is coherent. Pick an epimorphism $\mathcal{O}_U^n \to M|_U$. Then the kernel of

$$\varphi : \mathcal{O}_U^n \to M|_U \to N|_U$$

is of finite type since N is coherent. But the kernel of ψ is the image of the composite

$$\ker(\varphi) \to \mathcal{O}_U^n \to M|_U \ .$$

Thus by the first assertion including the addendum $\ker(\psi)$ is coherent. Note again that if M is only of finite type then we have shown that the kernel is also of finite type since it then admits a surjection from $\ker(\varphi)$.

To see that the cokernel of ψ is quasi-coherent we can by passing to the image assume without loss of generality that ψ is injective. Now observe that $\mathrm{coker}(\psi) = N/M$ is of finite type since it receives an epimorphism from N. Thus assume that we are given a morphism $g : \mathcal{O}_U^n \to \mathrm{coker}(\psi)$ and we need to show that $\ker(g)$ is of finite type. Since this morphism g is given by n-sections it can after further shrinking U be lifted to a morphism

$$g' : \mathcal{O}_U^n \to N$$

Then we get that $\ker(g)$ is the preimage of M under this map, i.e. we have a pullback square

$$\begin{tikzcd} \ker(g) \arrow[r] \arrow[d] & M \arrow[d] \\ \mathcal{O}_U^n \arrow[r] & N \end{tikzcd} .$$

Thus we have a short exact sequence

$$\ker(g) \to M \oplus \mathcal{O}_U^n \to N$$

which shows by the addendum to the second part that $\ker(g)$is of finite type.

Consider an extension with M_0 and M_2 coherent.

$$M_0 \to M_1 \to M_2$$

[3]This means: given a short exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of $\mathcal{O}_X$-module sheaves. Then if M_0 and M_2 are quasi-coherent, then so is M_1.

Then we note that any epimorphism $\mathcal{O}_U^n \to M_2$ locally lifts to a morphism $\mathcal{O}_U^n \to M_2$ together with an epimorphism $\mathcal{O}_U^m \to M_0$ the morphism

$$\mathcal{O}_U^m \oplus \mathcal{O}_U^n \to M_1$$

is then an epimorphism. Thus M_1 is of finite type. Now assume that we have a morphism

$$g : \mathcal{O}^n \to M_1$$

Then we consider the pullback

$$\begin{tikzcd} K \arrow[r] \arrow[d] & M_0 \arrow[d] \\ \mathcal{O}^n \arrow[r] & M_1 \end{tikzcd}$$

This is the kernel of $M_0 \oplus \mathcal{O}^n \to M_1$ and thus of finite type. But then the kernel of the morphism g is the kernel of $K \to M_0$ and thus also of finite type. $\square$

In particular since extensions are coherent, we see that direct sums of coherent modules are also coherent (which is in fact very easy to see directly from the definition).

Definition 4.10. *A module M over a ring R is called coherent if it is finitely generated and every finitely generated submodule is finitely presented. A ring R is called coherent, if it is coherent as a module over itself.*

Note that for a ring A to be coherent means that every finitely generated ideal is finitely presented.

Proposition 4.11. *An $\mathcal{O}_{\mathrm{Spec}(A)}$-module on $\mathrm{Spec}(A)$ is coherent precisely if it is of the form $\widetilde{M}$ for a coherent A-module M. If A is coherent then this is the case precisely if M is finitely presented.*

Proof. Since every coherent module is quasi-coherent it has to be of the form $\widetilde{M}$ for $M \in \mathrm{Mod}_A$. Now $\widetilde{M}$ is of finite type precisely if M is finitely generated (by Proposition 2.8). Assume M is coherent and we have a finitely generated submodule $N \subseteq M$. Then we have a map $A^n \to M$ which surjects onto N and whose kernel is the set of relations. This corresponds to a map $\mathcal{O}^n_{\mathrm{Spec}(A)} \to \widetilde{M}$ and therefore has finitely generated kernel, this N is finitely presented.
Conversely assume that M is coherent and assume we are given an open $U \subseteq \mathrm{Spec}(A)$ and a map

$$\varphi : \mathcal{O}_U^n \to \widetilde{M}|_U$$

We need to check that the kernel is of finite type. Since this can be done locally we can assume that $U = D(f)$. Then we need to verify that $M[1/f]$ is also coherent. to see that this map has a finitely generated kernel. But every finitely generated submodule of $M[1/f]$ is induced by a finitely genrated submodule of $N \subseteq M$ and thus is also finitely presented.
In general if a module is coherent then it is finitely presented, since we have a surjection $A^n \to M$ and the kernel is itself finitely presented. Conversely since cokernels of coherent modules are coherent (by Proposition 4.9 combined with the first part of the proof) we conclude that if M is finitely presented (i.e. the cokernel of a map $A^m \to A^n$) it is coherent if A is. $\square$

PROPOSITION 4.12. *On a noetherian scheme X a quasi-coherent sheaf is coherent precisely if it is of finite type.*[4].

PROOF. By definition coherence can be checked locally as well as finite type. Therefore we can immediately reduce to the affine case $X = \mathrm{Spec}(A)$. There the claim becomes that a module is coherent precisely if it is finitely generated. By definition, coherent modules are finitely generated. Conversely assume that M is finitely generated. We want to show that it is coherent. Thus assume we are given a map $R^n \to M$. Then the kernel is a submodule of R^n. Since R is noetherian every such submodule is finitely generated (this can be deduced by induction on n). Thus M is coherent. $\square$

EXAMPLE 4.13. Every noetherian ring is coherent since it certainly is finitely generated over itself.
On a noetherian scheme the structure sheaf $\mathcal{O}_X$ is coherent.

Recall that an algebra B over A is called finitely generated, if there are finitely many elements $b_1, ..., b_n$ such that the smallest A-subalgebra of B containing $b_1, ..., b_n$ is already all of B. This is equivalent to the assertion that there is a surjective map $A[x_1, ..., x_n] \to B$ of A-algebras, i.e. that B is a quotient of a polynomial ring. This does of course not imply that B is finitely generated as an A-module.

DEFINITION 4.14. *A morphism $f : Y \to X$ is of finite type, if it is quasi-compact, and there is an open cover of Y by $U_i = \mathrm{Spec}(B_i)$, such that $f|_{\mathrm{Spec}(B_i)}$ factors over some open $\mathrm{Spec}(A_i) \subseteq X$ and the induced map $A_i \to B_i$ exhibits B_i as a finitely generated A_i-algebra.*

Note that this definition could be read in two different ways: either one has to say that after choosing the B_i there have to exist A_i such that the map factors and then the B_i are of finite type over A_i or one can say that the A_i have to be choosen carefully so that the B_i are of finite type over them. We will see in the next proof that this is equivalent.

PROPOSITION 4.15. *A morphism of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type iff B is finitely generated as an A-algebra.*

PROOF. If B is finitely generated as an A-algebra then the corresponding morphism if of finite type. Now assume conversely that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type.
Conversely assume that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type. We choose a cover $\mathrm{Spec}(B_i)$ of $\mathrm{Spec}(B)$ with corresponding opens $\mathrm{Spec}(A_i)$ according to the definition. Since a localization of a ring at an element is of finite type we can refine the cover $\mathrm{Spec}(B_i)$ so that the B_i's are principal opens: $B_i = B[1/f_i]$. If we moreover have that the morphism $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)$ factors over some furher open $\mathrm{Spec}(A_i') \subseteq \mathrm{Spec}(A_i)$ then we have associated morophimsms

$$A_i \to A_i' \to B_i$$

and if B_i if finite over A_i then it is in particular finite over A_i. Using this observation we can refine the opens $\mathrm{Spec}(A_i)$ to also be principal (by also further shrinking the

[4]In fact, it would suffices that the scheme is locally noetherian, but we will not get into this here...

B_i's), i.e, $A_i = A[1/g_i]$. Thus we arrive at the situation that we have finitely generated algebras

$$B[1/f_i]$$

over $A[1/g_i]$. Now we see that since $B[1/f_i]$ is finite type over $A[1/g_i]$ it is also of finite type over A. Thus we reduce to the situation that $A_i = A$. Moreover since $\mathrm{Spec}(B)$ is quasi-compact we already can also reduce to a finite open cover. So finally we have the situation that we have a finite cover $f_1, ..., f_n$ of $\mathrm{Spec}(B)$ such that each $B[1/f_i]$ is a finitely generated A-algebra and we want to show that B is a finitely generated A-algebra.
Fix generators $b_{ij}/f_i^{n_{ij}}$ of $B[1/f_i]$ as an A-algebra. Then we claim that the collection of elements

$$(b_{ij}, f_i)$$

generated B as an A-algebra, i.e. that the morphism

$$A[B_{ij}, F_i] \to B$$

sending B_{ij} to b_{ij} and F_i to f_i is surjective. are surjective. This can be checked as a morphism of modules, which means that it suffices to verify that for each prime ideal $x \in \mathrm{Spec}(A[B_{ij}, F_i])$ we have that

$$A[B_{ij}, F_i]_x \to B_x$$

is surjective (since surjectivity for quasi-coherent sheaves can be checked at stalks). Now if $x \in D(F_i)$ for some i then we have that this is true since by assumption the morphisms

$$A[B_{ij}, F_i^{\pm}] \to B[f_i^{-1}]$$

are surjective. If x is not in any $D(F_i)$ then $B_x = 0$ since the $D(g_i)$ is a cover of $\mathrm{Spec}(B)$ (said differnetly: then x does not lie in the image of the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A[B_{ij}, F_i])$). $\square$

PROPOSITION 4.16. *If $f : Y \to X$ is a morphism of finite type and X is noetherian, then Y is noetherian.*

PROOF. Y is quasi-compact since the morphism is. For a given affine open $\mathrm{Spec}(B) \subseteq Y$ we find $\mathrm{Spec}(A) \subseteq X$ such that B is finitely generated as an A-algebra. Since X is noetherian we deduce that A is noetherian. But A finitely generated A algebra is then also noetherian by Hilbert's basis theorem. $\square$

5. Dimension

5.1. Integral and finite ring extensions.

DEFINITION 5.1. *Let $R \hookrightarrow R'$ be an inclusion of rings.*

(1) *$a \in R'$ is* integral *over R if there exists a monic polynomial $f \in R[x]$ such that $f(a) = 0$, i.e. we have a relation of the form*

$$a^n + c_{n-1}a^{n-1} + \ldots + c_0 = 0,$$

with $c_i \in R$.

(2) *$R \subseteq R'$ is an* integral extension *if all $a \in R'$ are integral over R.*

(3) *$R \subseteq R'$ is a* finite extension *if R' is finitely generated as R-module.*

More generally, we call a ring homomorphism $\phi : R \to R'$ integral (finite) if $\phi(R) \subseteq R'$ is an integral (finite) extension.

EXAMPLE 5.2. Suppose $R \subseteq R'$ is an inclusion of fields. Then R' is integral over R if and only if R' is an algebraic field extension of R, and is finite over R if and only if R' is a finite field extension of R.

EXAMPLE 5.3. Suppose R is a unique factorisation domain, and let R' be its field of fractions. Then $a \in R'$ is integral if and only if $a \in R$. Indeed, let $a = \frac{p}{q}$ with p, q coprime in R, and suppose

$$\left(\frac{p}{q}\right)^n + c_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + c_0 = 0,$$

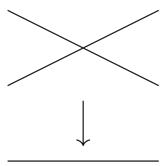
with $c_i \in R$. Multiplying with q^n, we see

$$p^n + c_{n-1}p^{n-1}q + \ldots + c_0 q^n = 0,$$

so $q | p^n$. Since p, q were assumed coprime, this implies that q is a unit, so $a = \frac{p}{q} \in R$.

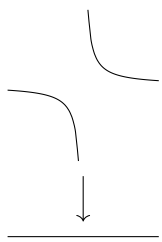
EXAMPLE 5.4. Let $R = \mathbb{C}[x]$, $R' = R[y]/f = \mathbb{C}[x, y]/f$, with $f \in R[y]$ nonconstant. (So $R \to R'$ is injective)

(1) Let $f = y^2 - x^2$. Then $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ looks like



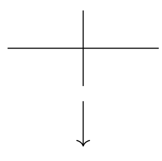
Observe that $R \subseteq R'$ is integral, since $\overline{y}$ satisfies $\overline{y}^2 - \overline{x}^2 = 0$. Geometrically, all fibers are finite and nonempty. [5]

(2) Let $f = xy - 1$.



Then it is easy to see that $\overline{y}$ is not integral over R (for example by applying Example 5.3). Here the fiber over the origin in $\mathrm{Spec}(R) = \mathbb{A}^1$ (i.e. the point given by the ideal (x)) is empty.

(3) For $f = xy$, we get



Again, $\overline{y}$ is not integral. Here the fibers are all nonempty, but the fiber over the origin in $\mathrm{Spec}(R) = \mathbb{A}^1$ is infinite.

THEOREM 5.5. *Let $R \subseteq R'$. Then $a \in R'$ is integral over R if and only if a is contained in an R-subalgebra of R' which is a finitely generated R-module.*

[5]Here we are using that it suffices to check this fo $\overline{y}$ since the product and sum of integral elements is again integral as we will see soon (Corollary 5.6).

PROOF. Suppose a is integral over R, i.e. we have

$$a^n + c_{n-1}a^{n-1} + \ldots + c_0 = 0,$$

with $c_i \in R$. Then the R-submodule of R' generated by $1, a, \ldots, a^{n-1}$ is closed under multiplication, thus an R-subalgebra of R'.
Conversely, assume $a \in S \subseteq R'$, and S is a subalgebra which is finitely generated as R-module. Pick generators $m_1, \ldots, m_n$ of S as R-module, and choose $b_{ij} \in R$ with

$$a \cdot m_i = \sum_{j=1}^{n} b_{ij} m_j$$

for all i. Let $B = (b_{ij})$ be the resulting $n \times n$ matrix, so we have

$$(a \cdot \mathrm{Id}_n - B) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0. \tag{11}$$

Recall from linear algebra that for every matrix M there is an adjugate matrix $\mathrm{adj}(M)$ with

$$\mathrm{adj}(M) \cdot M = \det(M) \cdot \mathrm{Id}_n .$$

This is usually done fields, but since the entries of $\mathrm{adj}(M)$ are polynomial expressions with integer coefficients in the entries of M, as is the determinant $\det(M)$, the above statement holds over any commutative ring. Multiplying (11) with $\mathrm{adj}(a \cdot \mathrm{Id}_n - B)$, we get

$$\det(a \cdot \mathrm{Id}_n - B) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Since $1 \in S$ is an R-linear combination of the generators m_i, this implies that $\det(a \cdot \mathrm{Id}_n - B) = 0$. This means that a is a zero of the characteristic polynomial of B, which is monic. □

COROLLARY 5.6. *For $R \subseteq R'$, the set of elements of R' which are integral over R is a subring of R'.*

PROOF. Write $\overline{R} = \{a \in R' \mid a \text{ integral over } R\}$. Then for any $a, b \in \overline{R}$ we find sub-R-algebras $A, B \subseteq R'$ which are finitely generated as R-modules and contain a, b. Writing $A \cdot B$ for the span of all products $x \cdot y$ with $x \in A$, $y \in B$, we see that $A \cdot B$ is also a subalgebra which is finitely generated as R-module. Since it contains $a \pm b$, $a \cdot b$, it follows that those elements are also integral over R. Thus $\overline{R}$ is a subring. □

COROLLARY 5.7. *If $R \subseteq R'$ is a finite extension, then it is also an integral extension.*

PROOF. Clear from the theorem. □

REMARK 5.8. The converse statement is false: Not every integral extension is finite. For example, take an infinite algebraic extension of fields, such as $\mathbb{Q} \subseteq \overline{\mathbb{Q}}$.

PROPOSITION 5.9. *Let $R \subseteq R' \subseteq R''$. If R' is integral (finite) over R and R'' is integral (finite) over R', then R'' is integral (finite) over R.*

PROOF. The statement for finite extensions is clear: If $a_1, \ldots, a_n$ are generators of R' as R-module, and $b_1, \ldots, b_m$ are generators of R'' as R'-module, then the $a_i b_j$ form generators of R'' as R-module. For the statement about integrality, let $a \in R''$ be any element. Then there exists a relation

$$a^n + c_{n-1}a^{n-1} + \ldots + c_0 = 0,$$

with $c_i \in R'$. So a is in fact integral over $R[c_0, \ldots, c_{n-1}] \subseteq R'$, and $R[c_0, \ldots, c_n, a]$ is finite over $R[c_0, \ldots, c_{n-1}]$. Since each of the c_i is integral over R, we can also see that $R[c_0, \ldots, c_{n-1}]$ is finite over R, so by the first statement, $R[c_0, \ldots, c_n, a]$ is finite over R. Thus, a is also integral over R, and since a was arbitrary, we see that R'' is integral over R. □

LEMMA 5.10. *Suppose $R \subseteq R'$ is an integral extension.*

(1) *If $I \subseteq R'$ is an ideal, then $R/(I \cap R) \hookrightarrow R'/I$ is an integral extension.*
(2) *If S is a subset of R, then*

$$R[S^{-1}] \hookrightarrow R'[S^{-1}]$$

is an integral extension.

PROOF. (1) Let $\overline{a} \in R'/I$. For some representative $a \in R'$, we can write a as zero of a monic polynomial with coefficients in R, so $\overline{a}$ is a zero of a monic polynomial with coefficients in $R/(I \cap R)$.
(2) Let $\frac{a}{s} \in R'[S^{-1}]$ with $a \in R'$. The we have a relation

$$a^n + c_{n-1}a^{n-1} + \ldots + c_n = 0, \quad c_i \in R.$$

Dividing by s^n, we find

$$\left(\frac{a}{s}\right)^n + \frac{c_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \ldots + \frac{c_n}{s^n} = 0,$$

with coefficients in $R[S^{-1}]$.

□

THEOREM 5.11 (Lying over). *Suppose $R \subseteq R'$ is an integral extension. Then for any prime ideal $\mathfrak{q} \subseteq R$, there exists $\mathfrak{q} \subseteq R'$ with $\mathfrak{p} \cap R = \mathfrak{q}$. Geometrically, this means that the map* $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ *is surjective.*

PROOF. We first discuss the case where R' is a field. Then we have to check that R is also a field. Let $b \neq 0 \in R$, and let $a = \frac{1}{b} \in R'$. Then we have

$$a^n + c_{n-1}a^{n-1} + \ldots + c_n = 0, \quad c_i \in R,$$

which, after multiplying with b^n and rearranging, gives

$$1 = b \cdot (-c_{n-1} + \ldots - c_0 b^{n-1}).$$

Here the term in parentheses is clearly in R, so we already have an inverse to b in R, and R is indeed a field.
For the general case, we may first localize to assume that $(R, \mathfrak{q})$ is local, since localisation preserve integral extensions by Lemma 5.10. Now we let $\mathfrak{p}$ be *any* maximal ideal of R', and observe that $R/(\mathfrak{p} \cap R) \to R'/\mathfrak{p}$ is an integral extension where the larger ring is a field. By the case we already did, thiss shows that $R/(\mathfrak{p} \cap R)$ is also a field, so $\mathfrak{p} \cap R$ is maximal. Since $(R, \mathfrak{q})$ was assumed local, $\mathfrak{p} \cap R = \mathfrak{q}$ is the unique maximal ideal, and the result follows. □

EXAMPLE 5.12. Lying over shows immediately that $\mathbb{C}[x] \to \mathbb{C}[x,y]/(xy-1)$ is not integral.

THEOREM 5.13 (Going up). *Suppose $\phi : R \to R'$ is an integral homomorphism. Let*

$$\mathfrak{q}_1 \subseteq \ldots \subseteq \mathfrak{q}_n$$

be a chain of prime ideals in R, and

$$\mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_m$$

be a chain of prime ideals in R', with $m < n$, such that $\phi^{-1}(\mathfrak{p}_i) = \mathfrak{q}_i$ for all $i \leq m$. Then we can extend the chain to $\mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_n$ with $\phi^{-1}(\mathfrak{p}_i) = \mathfrak{q}_i$ for all i.

PROOF. By induction, we can reduce to the case $m = 1$, $n = 2$. We thus have prime ideals $\mathfrak{q} \subseteq \mathfrak{q}'$ in R and $\mathfrak{p} \in R'$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{q}$, and want to find $\mathfrak{p} \subseteq \mathfrak{p}'$ with $\phi^{-1}(\mathfrak{p}') = \mathfrak{q}'$. By assumption, the map $R/\mathfrak{q} \to R'/\mathfrak{p}$ is injective and an integral extension, and $\mathfrak{q}'/\mathfrak{q}$ is a prime ideal of $R/\mathfrak{q}$. The claim now follows by applying the lying over theorem to this map. $\square$

EXAMPLE 5.14. Consider $\mathbb{C}[x] \subseteq \mathbb{C}[x,y]/(y^2 - x^2)$. In $\mathbb{C}[x]$, we have the chain of prime ideals $(0) \subseteq (x-1)$, which geometrically corresponds to the point $x = 1$ lying in the entire $\mathbb{A}^1$. In $\mathbb{C}[x,y]/(y^2 - x^2)$ we have the ideal $(x+y)$, whose intersection with $\mathbb{C}[x]$ is (0). Geometrically, $(x+y)$ is one of the two lines of which $\operatorname{Spec}(\mathbb{C}[x,y]/(y^2-x^2)$ is the union. The Going Up theorem guarantees that we find a prime ideal contained in $(x+y)$ whose intersection with $\mathbb{C}[x]$ is precisely $(x-1)$, i.e. that the selected line contains a point sitting over $x = 1$. This ideal can be given explicitly by $(x+y, x-1)$.

THEOREM 5.15 (Incomparability). *Let $R \subseteq R'$ be an integral ring extension. Suppose $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq R'$ are distinct prime ideals such that $\mathfrak{p}_1 \cap R = \mathfrak{p}_2 \cap R$. Then $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$.*

PROOF. Suppose $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Since they are assumed distinct, there is $a \in \mathfrak{p}_2$ with $a \notin \mathfrak{p}_1$. Since $R/(\mathfrak{p}_1 \cap R) \to R'/\mathfrak{p}_1$ is an integral extension, there is a relation

$$\overline{a}^n + \overline{c}_{n-1}\overline{a}^{n-1} + \ldots + \overline{c}_0 = 0 \tag{12}$$

in $R'/\mathfrak{p}_1$, with all $c_i \in R$. Pick a relation of minimal degree n. Since $a \in \mathfrak{p}_2$, we see that $\overline{c}_0 \in \mathfrak{p}_2/\mathfrak{p}_1 \subseteq R'/\mathfrak{p}_1$. But as $c_0 \in R$, $\overline{c}_0 \in (\mathfrak{p}_2 \cap R)/(\mathfrak{p}_1 \cap R) = 0$. So the constant term of (12) vanishes. But then we can divide by $\overline{a}$ to get a relation of smaller degree, contradicting our choice. $\square$

COROLLARY 5.16. *Let $R \subseteq R'$ be an integral extension.*

(1) *If R, R' are integral domains, then R is a field if and only if R' is a field.*
(2) *$\mathfrak{p}' \subseteq R'$ is maximal if and only if $\mathfrak{p}' \cap R \subseteq R$ is maximal.*

PROOF. That if R' is a field, then R is a field, we already saw in the proof of the Lying over theorem. For the other direction, assume R is a field and let $\mathfrak{p}' \subseteq R'$ be any maximal ideal. Since R' is an integral domain, (0) is a prime ideal, and we have $(0) \subseteq \mathfrak{p}'$. But $\mathfrak{p}' \cap R = 0 = (0) \cap R$, so by incomparability,$\mathfrak{p}' = (0)$ and thus R' is also a field.
The second statement reduces to the first by looking at the integral extension $R/(\mathfrak{p}' \cap R) \to R'/\mathfrak{p}'$. $\square$

EXAMPLE 5.17. $\mathbb{C}[x] \to \mathbb{C}[x,y]/(xy)$ is not integral: We have a nontrivial inclusion of prime ideals $(x) \subsetneq (x, y-1)$, but the intersection of both with $\mathbb{C}[x]$ is just (x). Similarly, $\mathbb{C}[x] \to \mathbb{C}[x,y]$ is also not integral.

5.2. Noether normalisation.

THEOREM 5.18. *Let R be a finitely generated algebra over a field k. Then there exists an injective map of k-algebras*

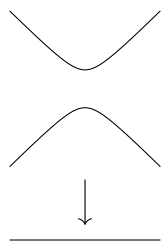
$$k[z_1, \ldots, z_d] \hookrightarrow R,$$

such that R is a finite extension of $k[z_1, \ldots, z_d]$.

EXAMPLE 5.19. $R = \mathbb{C}[x_1, x_2]/(x_1x_2 - 1)$. Then R is *not* finite over $\mathbb{C}[x_1]$, as we saw from the Lying Over theorem. We can fix this by changing coordinates: Let $x_1 = y_1 + y_2$, $x_2 = y_2 - y_1$, then R is isomorphic to $\mathbb{C}[y_1, y_2]/(y_2^2 - y_1^2 - 1)$. The extension

$$\mathbb{C}[y_1] \subseteq \mathbb{C}[y_1, y_2]/(y_2^2 - y_1^2 - 1)$$

is finite. Geometrically, in our new coordinates we have changed the direction of projection, and have fixed the problem of some fibers being empty.



PROOF OF THEOREM 5.18. By assumption, R is finitely generated as k-algebra. We perform induction on the number of generators $x_1, \ldots, x_n$. If $n = 0$, $R = k$ and the statement is trivial. So suppose $n > 0$. If the generators $x_1, \ldots, x_n$ are algebraically independent, i.e. the surjective map

$$k[x_1, \ldots, x_n] \to R$$

has trivial kernel, then we are done. So assume we have an element f in the kernel, i.e. a polynomial f in n variables such that $f(x_1, \ldots, x_n) = 0$ in R.
For suitable $a_1, \ldots, a_{n-1} \in \mathbb{N}$ (to be specified later), define new generators

$$y_1 = x_1 - x_n^{a_1}, \ldots, y_{n-1} = x_{n-1} - x_n^{a_{n-1}}, y_n = x_n.$$

These are still generators, since $x_n = y_n$ and $x_i = y_i + y_n^{a_i}$ for $i < n$. Expressing the relation f in terms of the new generators, we get

$$f(y_1 + y_n^{a_1}, \ldots, y_{n-1} + y_n^{a_{n-1}}, y_n) = f(x_1, \ldots, x_n) = 0.$$

If we look at the left hand side as an abstract polynomial in $y_1, \ldots, y_n$, every monomial $c \cdot \prod_i x_i^{k_i}$ in the original polynomial in the x_i gives

$$c \cdot y_n^{k_n}(y_{n-1} + y_n^{a_{n-1}})^{k_{n-1}} \cdots (y_1 + y_n^{a_1})^{k_1}.$$

If we expand this, the summand with largest degree in y_n is given by $c \cdot y_n^{k_n + a_{n-1}k_{n-1} + \ldots + a_1k_1}$.
We now choose $r \in \mathbb{N}$ larger than all degrees of monomials in the original $f(x_1, \ldots, x_n)$, and let $a_n = 1, a_{n-1} = r, \ldots, a_1 = r^{n-1}$. Then the numbers

$$k_n + rk_{n-1} + \ldots + r^{n-1}k_1$$

are different for each tuple $(k_1, \ldots, k_n)$ (this is basically the statement that every natural number can be written in a unique way in base r). So the contributions of different monomials cannot cancel, and we see that the monomial of largest y_n-degree in $f(y_1 + y_n^{a_1}, \ldots, y_{n-1} + y_n^{a_{n-1}}, y_n)$ is of the form $c \cdot y_n^k$ for some $c \neq 0$ and some k. Dividing by c, this shows that y_n is *integral* over the k-subalgebra $S \subseteq R$ generated by $y_1, \dots, y_{n-1}$. In particular, R is finite over S. Since S is generated by fewer generators, we know by induction that it is finite over some polynomial subalgebra $k[z_1, \dots, z_d]$, so R is also finite over $k[z_1, \dots, z_d]$. □

5.3. Dimension.

Definition 5.20. *Let R be a ring. The (Krull) dimension of R is defined as:*

$$\dim R := \sup_n \{\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n \mid \mathfrak{p}_i \subsetneq R \text{ prime}\}$$

Example 5.21. If R is a field, $\dim R = 0$. If $R = \prod_i k_i$, where the k_i are fields, then $\dim R = 0$. If $R = k[\varepsilon]/\varepsilon^2$ for a field k, then $\dim R = 0$.

Example 5.22. If R is a PID, $\dim R \leq 1$. Moreover, if R is not a field, $\dim R = 1$. For example $\dim \mathbb{Z} = 1$.

Proof. Suppose R is not a field. Then (0) is not a maximal ideal, and so for any maximal ideal $(0) \subsetneq \mathfrak{m}$ is a chain of length 1, so $\dim R \geq 1$. Conversely, suppose we have a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Without limiting generality, we may assume $\mathfrak{p}_0 = (0)$ and $\mathfrak{p}_1 \neq (0)$. Write $\mathfrak{p}_1 = (p_1)$ and $\mathfrak{p}_2 = (p_2)$ for irreducible elements p_1, p_2. Then $p_1 \in \mathfrak{p}_2$ shows that p_2 divides p_1, so they are in fact unit multiples of each other and $\mathfrak{p}_1 = \mathfrak{p}_2$, contradiction. So $\dim R \leq 1$. □

Lemma 5.23. *Let $R \hookrightarrow R'$ be an integral extension. Then* $\dim R = \dim R'$.

Proof. First we show $\dim R \leq \dim R'$. Consider a chain of prime ideals $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ in R. By the lying over theorem, we find a prime ideal $\mathfrak{q}_0 \subseteq R'$ with $\mathfrak{q}_0 \cap R = \mathfrak{p}_0$. By the going up theorem, we can extend to a sequence of primes $\mathfrak{q}_0 \subsetneq \dots \subsetneq \mathfrak{q}_n$ with $\mathfrak{q}_i \cap R = \mathfrak{p}_i$. So $n \leq \dim R'$, and thus $\dim R \leq \dim R'$.
For the other direction, let $\mathfrak{q}_0 \subsetneq \dots \subsetneq \mathfrak{q}_n$ be a chain of prime ideals in R'. Let $\mathfrak{p}_i = \mathfrak{q}_i \cap R$. By the incomparability theorem, the $\mathfrak{p}_i$ are distinct, and so

$$\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$$

shows $\dim R \geq n$, thus $\dim R \geq \dim R'$. □

Example 5.24. Let $f(x) \in \mathbb{Z}[x]$ be a monic non-constant polynomial. Then $\mathbb{Z} \to \mathbb{Z}[x]/f$ is integral, so $\dim \mathbb{Z}[x]/f = \dim \mathbb{Z} = 1$.

Example 5.25. Let k be a field. Then $\dim k[x] = 1$ since $k[x]$ is a PID. Similarly, we expect $\dim \mathbb{Z}[x] = \dim \mathbb{Z} + 1 = 2$, but this is more subtle.

Proof. The chain $(0) \subsetneq (p) \subsetneq (p, x)$ shows $\dim \mathbb{Z}[x] \geq 2$. To see $\dim \mathbb{Z}[x] \leq 2$, consider a chain $(0) \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3$. Letting f be some irreducible element in $\mathfrak{p}_1$, i.e. $(f) \subseteq \mathfrak{p}_1$, we may further replace $\mathfrak{p}_1$ by (f) and assume it is principal. Now there are two cases:

(1) If $f = p$ is a constant polynomial, then mod f we get $(0) \subsetneq \overline{\mathfrak{p}_2} \subsetneq \overline{\mathfrak{p}_3}$, a chain of length 2 in $\mathbb{F}_p[x]$. But $\dim \mathbb{F}_p[x] = 1$, contradiction.
(2) If $f = c_n x^n + \dots + c_0 \in \mathbb{Z}[x]$ is a non-constant irreducible polynomial, then $\mathbb{Z} \to \mathbb{Z}[x]/f$ is injective. Again, it would suffice to show that $\dim \mathbb{Z}[x]/f = 1$, or equivalently, that for *any* nonzero prime ideal $\mathfrak{p}$ in $\mathbb{Z}[x]/f$, the quotient $(\mathbb{Z}[x]/f)/\mathfrak{p}$ is a field. Consider the integral closure R of $\mathbb{Z}$ in $\mathbb{Z}[x]/f$, i.e.

the subring of $\mathbb{Z}[x]/f$ of all elements which are integral over $\mathbb{Z}$. Multiplying f with c_n^{n-1}, we see

$$(c_n x)^n + c_{n-1}(c_n x)^{n-1} + \ldots + c_0 c_n^{n-1} = 0 \text{ in } \mathbb{Z}[x]/f,$$

so $c_n x$ is integral over $\mathbb{Z}$, i.e. $c_n x \in R$. Since x generates $\mathbb{Z}[x]/f$, this shows that for any $z \in \mathbb{Z}[x]/f$, $c_n^N f \in R$ for large enough N.

It remains to show that for any nonzero prime ideal $\mathfrak{p} \subseteq \mathbb{Z}[x]/f$, the quotient $(\mathbb{Z}[x]/f)/\mathfrak{p}$ is a field. First observe that $\mathfrak{p} \cap R \neq 0$, since for $z \in \mathfrak{p}$ nonzero, some $c_n^N z \in \mathfrak{p} \cap R$, and $\mathbb{Z}[x]/f$ is a domain, so $c_n^N \neq 0$. But since R is integral over $\mathbb{Z}$, we have $\dim R = \dim \mathbb{Z} = 1$, and so $R/(\mathfrak{p} \cap R)$ is a field, necessarily of positive characteristic, since it is in particular integral over $\mathbb{Z}$. So there is a prime number $\ell \in \mathbb{Z}$ which is zero in $R/(\mathfrak{p} \cap R)$, so also $\ell = 0$ in $(\mathbb{Z}[x]/f)/\mathfrak{p}$. It follows that we have a surjective map

$$(\mathbb{Z}/\ell)[x] \to (\mathbb{Z}[x]/f)/\mathfrak{p}.$$

Since $\mathbb{Z}/\ell[x] = \mathbb{F}_\ell[x]$ is a PID, it has dimension 1, and so a domain which is a quotient of it is automatically a field.

□

More generally, we would like to show that $\dim k[x_1, \ldots, x_n] = n$ for a field k.

THEOREM 5.26. *Let R be a finitely generated k-algebra, and let $k[x_1, \ldots, x_n] \hookrightarrow R$ be a finite extension (e.g. as in Noether normalisation). Then* $\dim R = n$.

For the proof, we will require the notion of *transcendence bases.*

DEFINITION 5.27. *Let $K \subseteq L$ be a field extension. We call $\alpha_1, \ldots, \alpha_n \in L$* algebraically dependent *over K if there exists nonzero $f \in K[x_1, \ldots, x_n]$ with $f(\alpha_1, \ldots, \alpha_n) = 0$. Otherwise, $\alpha_1, \ldots, \alpha_n$ are* algebraically independent *over K.*
$(\alpha_1, \ldots, \alpha_n)$ form a transcendence basis *over K if $\alpha_1, \ldots, \alpha_n$ are algebraically independent over K and L is algebraic over $K(\alpha_1, \ldots, \alpha_n)$.*

LEMMA 5.28. *For $\alpha_1, \ldots, \alpha_n \in L$ algebraically independent over K, we have that the injection*

$$K[x_1, \ldots, x_n] \to K(\alpha_1, \ldots, \alpha_n), f \mapsto f(\alpha_1, \ldots, \alpha_n),$$

extends to an isomorphism $\mathrm{Frac}(K[x_1, \ldots, x_n]) \cong K(\alpha_1, \ldots, \alpha_n)$.

PROOF. The map is injective by the definition of algebraic independence, so it factors through the fraction field. The resulting map is surjective (because it takes the x_i to field generators α_i), and injective because it is a field homomorphism. □

This justifies us to write $K(x_1, \ldots, x_n)$ for the fraction field of $K[x_1, \ldots, x_n]$.

LEMMA 5.29. *Let $\alpha_1, \ldots, \alpha_n \in L$ be algebraically independent over K, and let $\beta \in L$. Then $\alpha_1, \ldots, \alpha_n, \beta$ are algebraically dependent over K if and only if β is algebraic over $K(\alpha_1, \ldots, \alpha_n)$.*

PROOF. Suppose $\alpha_1, \ldots, \alpha_n, \beta$ are algebraically dependent. Then there exists a nonconstant polynomial f with $f(\alpha_1, \ldots, \alpha_n, \beta) = 0$. Consider the polynomial

$$q(Y) = f(\alpha_1, \ldots, \alpha_n, y) \in K(\alpha_1, \ldots, \alpha_n)[y].$$

Then q is nonconstant and $q(\beta) = 0$, so β is algebraic over $K(\alpha_1, \ldots, \alpha_n)$.

For the other direction, assume β is a zero of a polynomial

$$c_d y^d + \ldots + c_0 \in K(\alpha_1, \ldots, \alpha_n)[y] \cong K(x_1, \ldots, x_n)[y].$$

Then we can clear denominators to obtain a multiple

$$f = c_d' y^d + \ldots + c_0' \in K[x_1, \ldots, x_n, y].$$

This is now a nonzero polynomial with $f(\alpha_1, \ldots, \alpha_n, \beta) = 0$. $\square$

Lemma 5.30 (Swapping Lemma). *Let $\alpha_1, \ldots, \alpha_n \in L$ be a transcendence basis of L over K, and let $\beta \in L$ be non-algebraic over K. Then there exists i such that*

$$\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_{i+1}, \ldots, \alpha_n$$

also forms a transcendence basis of L/K. Moreover, if $\beta, \alpha_1, \ldots, \alpha_j$ are algebraically independent over K, we may pick i from $j+1, \ldots, n$.

(Note the similarity to the Steinitz exchange lemma in linear algebra.)

Proof. By assumption (and the previous lemma), we have that $\alpha_1, \ldots, \alpha_n, \beta$ algebraically dependent over K. Suppose that $\beta, \alpha_1, \ldots, \alpha_j$ are algebraically independent over K, but $\beta, \alpha_1, \ldots, \alpha_{j+1}$ are algebraically dependent. Then we claim that swapping β with α_{j+1} gives also a transcendence basis. Indeed, let

$$K' = K(\alpha_1, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n).$$

By the lemma, α_{j+1} is algebraic over $K(\alpha_1, \ldots, \alpha_{j-1}, \beta)$, so in particular the extensions $K' \subseteq K'(\alpha_{j+1}) \subseteq L$ are algebraic. It remains to show that $\alpha_1, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n$ are algebraically independent. Suppose not, then we would have that β is algebraic over $K'' = K(\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)$, and then $K'' \subseteq K''(\beta) = K' \subseteq K'(\alpha_{j+1})$ would be algebraic extensions. But then α_{j+1} would be algebraic over K'', which contradicts the original assumption that $\alpha_1, \ldots, \alpha_n$ are algebraically independent. $\square$

Corollary 5.31. *If $\alpha_1, \ldots, \alpha_n$ are a transcendence basis for L/K, and $\beta_1, \ldots, \beta_m$ are algebraically independent over K, then $m \leq n$.*

Proof. Suppose $m > n$. By the swapping lemma, we can inductively replace entries of the original transcendence basis by $\beta_1, \ldots, \beta_n$. We obtain that $\beta_1, \ldots, \beta_n$ is a transcendence basis. But then we get that $\beta_1, \ldots, \beta_n, \beta_{n+1}$ are not algebraically independent, contrary to the assumption. $\square$

Corollary 5.32. *If $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ are transcendence bases of L/K, they have the same length $n = m$.*

Proof. By the above, $n \leq m$ and $m \leq n$. $\square$

Definition 5.33. *Suppose $K \leq L$ has a transcendence basis $\alpha_1, \ldots, \alpha_n$. Then we define the transcendence degree* $\operatorname{trdeg} L/K = n$. *Otherwise, we define* $\operatorname{trdeg} L/K = \infty$.

Lemma 5.34. *Let K be a field, and let $R = k[x_1, \ldots, x_n]/f$ for f irreducible. Then* $\operatorname{Frac}(R)$ *has transcendence degree $n-1$ over K.*

Proof. Since f is nonconstant, at least one variable appears nontrivially in f, without limiting generality we may assume x_n does. We claim that $k[x_1, \ldots, x_{n-1}] \to R$ is injective. Indeed, if not, there is an element g in the kernel $(f) \cap k[x_1, \ldots, x_{n-1}]$, but this is then a polynomial in $x_1, \ldots, x_{n-1}$ which is a multiple of f, which is absurd.

It follows that $\overline{x}_1, \ldots, \overline{x}_{n-1}$ are algebraically independent in $\mathrm{Frac}(R)$. Moreover, $\overline{x}_1, \ldots, \overline{x}_{n-1}$ are algebraically dependent in $\mathrm{Frac}(R)$ by definition, and generate, so $\overline{x}_1, \ldots, \overline{x}_{n-1}$ form a transcendence basis for $\mathrm{Frac}(R)/K$. $\square$

THEOREM 5.35. *Let A be a finitely generated k-algebra and integral domain, where k is a field. Then* $\operatorname{trdeg} \mathrm{Frac}(A)/k < \infty$ *(i.e.: there exists a transcendence basis), and* $\operatorname{trdeg} \mathrm{Frac}(A)/k = \dim(A)$.

PROOF. Given A, we choose a Noether normalisation $k[x_1, \ldots, x_d] \hookrightarrow A$. Then since $k[x_1, \ldots, x_d] \subseteq A$ is a finite extension, $\operatorname{trdeg} \mathrm{Frac}(A)/k = \operatorname{trdeg} k(x_1, \ldots, x_d)/k = d$, so in particular, it is finite.
Now we proceed by induction on trdeg. Suppose $\operatorname{trdeg} \mathrm{Frac}(A)/k = 0$. Then since $\mathrm{Frac}(A)/k$ is an algebraic extension, A/k is an integral extension (in fact, a finite extension, since it is finitely generated and each generator is algebraic over k). But we saw that integral extensions of fields are fields, so A is a field and $\dim A = 0$.
Now suppose by induction that if trdeg $< n$, then trdeg = dim. Choosing a normalisation, we may assume $A = k[x_1, \ldots, x_n]$, since neither trdeg nor dim change under finite extensions. We claim that $\dim A = n$. The chain of prime ideals

$$(0) \subsetneq (x_1) \subsetneq \ldots \subsetneq (x_1, \ldots, x_n)$$

shows that $\dim(A) \geq n$. Suppose we are given an arbitrary chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m.$$

Then we may assume $\mathfrak{p}_0 = (0)$, and by replacing $\mathfrak{p}_1$ by (f) for an irreducible element $f \in \mathfrak{p}_1$, also that $\mathfrak{p}_1 = (f)$. By the above lemma, we have $\operatorname{trdeg} \mathrm{Frac}(k[x_1, \ldots, x_n]/f) = n - 1$, so by induction, $\dim k[x_1, \ldots, x_n]/f = n - 1$. So the remaining chain

$$(0) \subsetneq \mathfrak{p}_2/(f) \subsetneq \ldots \subsetneq \mathfrak{p}_m/(f)$$

proves that $m - 1 \leq n - 1$, thus $m \leq n$ and $\dim A = n$. $\square$

COROLLARY 5.36. *Let A be a finitely generated k-algebra. Then for any finite extension $k[x_1, \ldots, x_n] \subseteq A$, we have* $\dim A = n$.

PROOF. We know $\dim A = \dim k[x_1, \ldots, x_n] = \operatorname{trdeg} k(x_1, \ldots, x_n)/k = n$. $\square$

It is instructive to see that this vastly generalizes Lemma 3.10, which we used last semester to prove the Nullstellensatz:

COROLLARY 5.37 (Zariski-Lemma / Nullstellensatz). *Assume that K is a finitely generated k-algebra which is a field. Then K/k is algebraic.*

PROOF. We have $\operatorname{trdeg} K/k = \dim K = 0$, so K is an algebraic extension of k. $\square$

5.4. Dimension in topology.

DEFINITION 5.38. *Let X be a topological space. A* chain *of irreducible closed subsets in X is a sequence $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$, where each Z_i is closed and irreducible. Then*

$$\dim(X) := \sup_n \{Z_0 \subsetneq \ldots \subsetneq Z_n \text{ chains of irreducible closed subsets}\}$$

if X is nonempty, and $\dim(\emptyset) := -\infty$ *by definition.*

REMARK 5.39. Suppose X is sober, i.e. every irreducible closed subset $Z \subseteq X$ is given by $\overline{\{z\}}$ for a unique point $z \in Z$. (Recall that $\mathrm{Spec}(A)$ is sober for each ring A.) Then every chain $Z_0 \subsetneq \ldots \subsetneq Z_n$ corresponds to a sequence of points $z_0, \ldots, z_n$ where $z_i \in \overline{\{z_{i+1}\}}$ for each $i = 0, \ldots, n-1$.
We say that x is a *specialisation* of y if $x \in \overline{\{y\}}$, written $y \rightsquigarrow x$, so we may also write

$$\dim X = \sup_n \{x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n \text{ specialisation chains}\}.$$

REMARK 5.40. (1) If $X = \bigcup X_i$ for X_i irreducible components of X, we see that $\dim(X) = \sup \dim(X_i)$.
(2) If $W \subseteq X$ is any subspace, then $\dim W \subseteq \dim X$.

PROOF. The first claim follows from the fact that any chain is contained in one of the X_i by irreducibility. For the second claim, observe that for any chain $Z_0 \subsetneq \ldots \subsetneq Z_n$ in W, the closures

$$\overline{Z_0} \subseteq \ldots \subseteq \overline{Z_n}$$

are all distinct, closed and irreducible, thus form a chain in X and show $\dim X \geq n$, thus $\dim X \geq \dim W$. Indeed, note that $\overline{Z_i} \cap W = Z_i$, so they are distinct. To see that $\overline{Z_i}$ is irreducible, assume $\overline{Z_i} = A \cup B$ with both closed. Then one of the intersections $A \cap W, B \cap W$ has to agree with Z_i by irreducibility of Z_i, say A. Thus $Z_i \subseteq A$, and therefore also $\overline{Z_i} \subseteq A$, thus $\overline{Z_i}$ is irreducible. $\square$

EXAMPLE 5.41. For a ring R, $\dim \mathrm{Spec}\, R = \dim R$ in view of the (order-reversing) bijection between prime ideals $\mathfrak{p} \subseteq R$ and closed irreducible subsets $V(\mathfrak{p}) \subseteq \mathrm{Spec}(R)$.

REMARK 5.42. For $U \subseteq X$ an open subspace, there is a *bijection* between irreducible closed subsets of U and irreducible closed subsets of X that meet U, given by taking closure in X / intersecting with U. It follows that a scheme X has dimension $\dim X = n$ iff it has an affine open cover $\bigcup \mathrm{Spec}(A_i)$ with $\dim \mathrm{Spec}(A_i) \leq n$ for all i and $\dim \mathrm{Spec}(A_i) = n$ for at least one i.

6. Varieties

In this section, we want to define varieties, which are a nice subclass of schemes over an algebraically closed field k. The main idea will be that the category of affine varieties is equivalent to the category of affine algebraic sets we defined in the beginning of the first lecture. So one should think of varieties as a good non-affine generalisation of affine algebraic sets.We start by explaining the concept of separatedness, which is crucial for the the definition of varieties.
Recall that a topological space X is called Hausdorff if points can be separated by open sets, i.e. if $x \neq y$, we find neighbourhoods $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$. This condition is important in geometry: Manifolds are always required to be Hausdorff, excluding example such as $\mathbb{R}^n \amalg_{\mathbb{R}^n \setminus \{0\}} \mathbb{R}^n$.
We want a similar property for schemes, satisfied by $\mathbb{A}^n$ and $\mathbb{P}^n$, but for example not by $\mathbb{A}^1_k \amalg_{\mathbb{A}^1_k \setminus \text{“}0\text{”}} \mathbb{A}^1_k$, i.e. the pushout of $\mathrm{Spec}\, k[x]$ and $\mathrm{Spec}\, k[x]$ along $\mathrm{Spec}\, k[x^{\pm 1}]$ where both maps are the same (this is different from the pushout defining $\mathbb{P}^1$!). Hausdorff is not a good property, since $\mathrm{Spec}(A)$ is only Hausdorff if A is 0-dimensional.
However, recall the following:

PROPOSITION 6.1. *A topological space X is Hausdorff iff the diagonal subspace $\Delta \subseteq X \times X$ is a closed subspace.*

PROOF. Δ is closed if and only if its complement is open, i.e. if every point has an open neighbourhood. Since the product topology is generated by open boxes, this explicitly means that every $(x, y) \in X \times X$ with $x \neq y$ lies in a subset of the form $U \times V \subseteq X \times X \setminus \Delta$, where $U, V \subseteq X$ are open subsets. But $U \times V$ being disjoint from Δ means precisely that $U \cap V = \emptyset$, so this property is equivalent to Hausdorff. □

It turns out this property does generalize well to schemes:

DEFINITION 6.2. *A morphism of schemes $f : X \to S$ is called separated if the diagonal map*

$$\Delta_{X/S} : X \to X \times_S X$$

is a closed immersion. A scheme X is called separated if $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Note that unlike for topological spaces, this is *not* the same as asking for the underlying space to be Hausdorff, since the underlying space of $X \times_S X$ is not the pullback of spaces.
To get a feeling for this condition, we introduce a weaker notion of closed immersions:

DEFINITION 6.3. *A morphism $i : X \to Y$ of schemes is called a* locally closed immersion *if:*

(1) *The underlying map of spaces is a homeomorphism onto its image, and as spaces, $i(X) \subseteq Y$ is open in its closure $\overline{i(X)}$.*
(2) *The map $i^\sharp : i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an epimorphism of sheaves.*

Note that the condition is in terms of the map $i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, not the map $\mathcal{O}_Y \to i_*\mathcal{O}_X$ of sheaves on Y.

EXAMPLE 6.4.
(1) Every closed immersion is in particular a locally closed immersion. Indeed, i^{-1} preserves stalks, and for a closed immersion we also have $i^{-1} \circ i_* = \mathrm{id}$, so if $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is an epimorphism, also $i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an epimorphism.
(2) Every open immersion (inclusion of an open subscheme) is also a locally closed immersion: Indeed, an open immersion can be characterized as being a map $i : U \to X$ which is a homeomorphism on an open subspace, such that the map $i^\sharp : i^{-1}\mathcal{O}_X \to \mathcal{O}_U$ is an *isomorphism*, in particular surjective.

LEMMA 6.5. *If a locally closed immersion has closed image, it is a closed immersion.*

PROOF. Let $i : X \to Y$ be a locally closed immersion with closed image. We only have to check that $i^\sharp : \mathcal{O}_Y \to i_*\mathcal{O}_X$ is an epimorphism. We can check this on stalks, and since $i(X)$ is closed, we have $(i_*\mathcal{O}_X)_y = 0$ for each $y \notin i(X)$. Now observe that $i^{-1}i_*\mathcal{O}_X \cong \mathcal{O}_X$ since i is a homeomorphism onto its image, and since $(i^{-1}\mathcal{F})_x \cong (\mathcal{F})_{i(x)}$, surjectivity on stalks at points in the image $i(X)$ is equivalent to surjectivity of $i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. □

PROPOSITION 6.6. *A morphism $i : X \to Y$ is a locally closed immersion if and only if it can be written as a closed immersion followed by an open immersion.*

PROOF. Since closed and open immersions are both in particular locally closed immersions, and locally closed immersions can be composed, one direction is clear. For the other, assume $i : X \to Y$ is a locally closed immersion. Then the subspace $i(X)$ is open in $\overline{i(X)}$, so by the definition of the subspace topology, there exists an open $U \subseteq Y$ such that $i(X) = \overline{i(X)} \cap U$. So $i(X)$ is closed in U. Regarding U as an open subscheme, we get the desired factorisation

$$X \to U \to Y,$$

where the first one is automatically a closed immersion since its image is closed. $\square$

This suggests a nice way to thinking about locally closed immersions: They are precisely those maps $i : X \to Y$ such that each point in the image $i(X)$ admits a neighbourhood $U \subseteq Y$ for which $i^{-1}(U) \to U$ is a closed immersion. The difference to closed immersions (which we can also check locally) is that there we require such a neighbourhood around *each* point of Y, not just in $i(X)$.
We will now see that the sheaf part of separatedness is automatic.

PROPOSITION 6.7. *Let $f : X \to S$ be a morphism of schemes. Then the diagonal map $\Delta_{X/S} : X \to X \times_S X$ is a locally closed immersion. If X and S are affine, then this is in fact just a closed immersion.*

PROOF. First assume that $X = \operatorname{Spec}(A)$, $S = \operatorname{Spec}(R)$ are affine. Then we have

$$X \times_S X \cong \operatorname{Spec}(A \otimes_R A),$$

and the diagonal map corresponds to the multiplication map $A \otimes_R A \to A$. This is clearly surjective, to it is a closed immersion. In general, for $x \in X$, pick an affine neighbourhood $U = \operatorname{Spec}(A) \subseteq X$ of x mapping to an affine open $\operatorname{Spec}(R) \subseteq S$. Then in $X \times_S X$, $W = \operatorname{Spec}(A) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(A) \cong \operatorname{Spec}(A \otimes_R A)$ is an open neighbourhood of $\Delta_{X/S}(x)$. Its preimage is given by

$$\Delta^{-1}(W) \cong \operatorname{Spec}(A),$$

so it suffices to check that $\operatorname{Spec}(A) \to \operatorname{Spec}(A \otimes_R A)$ is a closed immersion, which we know from the affine case. $\square$

COROLLARY 6.8. *A morphism $X \to S$ of affine schemes is automatically separated. In particular, every affine scheme X is separated.*

COROLLARY 6.9. *A morphism $f : X \to S$ is separated if and only if the subspace $\Delta_{X/S}(X) \subseteq X \times X$ is closed.*

PROPOSITION 6.10. *For a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z, \end{tikzcd}$$

we have

(1) *If f and g are separated, so is h.*
(2) *If h is separated, so is f.*

PROOF. We have a diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Y}"] \arrow[dr, "\Delta_{X/Z}"'] & (X \times_Z X) \times_{(Y \times_Z Y)} Y \arrow[d, "\mathrm{id} \times \Delta_{Y/Z}"] \\ & X \times_Z X \cong (X \times_Z X) \times_{(Y \times_Z Y)} (Y \times_Z Y), \end{tikzcd}$$

If f and g are separated, the horizontal map is a closed immersion, the vertical map is a pullback of a closed immersion, so the composite is also a closed immersion. If h is a closed immersion, we simply observe that on spaces, the image of the horizontal map is the preimage of the image of the diagonal map under the vertical map. So the image of the horizontal map is also closed, thus f is separated. $\square$

Recall that we previously defined a scheme X to be separated if $X \to \mathrm{Spec}(\mathbb{Z})$ is a separated morphism. If we are instead working in $\mathrm{Sch}_{\mathrm{Spec}(k)}$, we might instead want to call a scheme X over $\mathrm{Spec}(k)$ separated if $X \to \mathrm{Spec}(k)$ is separated. Since the morphism of affines $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Z})$ is automatically separated, the previous proposition shows that $X \to \mathrm{Spec}(k)$ is separated if and only if $X \to \mathrm{Spec}(\mathbb{Z})$ is separated, so the two notions agree.
Finally, we give a useful criterion to check whether a scheme is separated.

Proposition 6.11.
(1) *If X is a separated scheme, then for every pair $U, V \subseteq X$ of affine opens, the intersection $U \cap V$ is also affine, and furthermore the map*
$$\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$$
is surjective.
(2) *If a scheme X admits a cover by affine opens U_i, such that $U_i \cap U_j$ is affine for each i, j, and furthermore the maps*
$$\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$
are surjective, then X is separated.

Proof. For the first statement, observe that the preimage of $U \times V$ under $\Delta_X : X \to X \times X$ is $U \cap V$. So Δ restricts to a closed immersion
$$U \cap V \to U \times V.$$
If $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$, then this shows that $U \cap V \cong \mathrm{Spec}((A \otimes B)/I)$ for some ideal I.
For the converse, observe that the $U_i \times U_j$ cover $X \times X$. So to check whether $\Delta : X \to X \times X$ is a closed immersion, it suffices to check whether $U_i \cap U_j \to U_i \times U_j$ is a closed immersion for all i, j. Again, if $U_i = \mathrm{Spec}(A_i)$ and $U_j = \mathrm{Spec}(A_j)$, this exactly means that $U_i \cap U_j \cong \mathrm{Spec}((A_i \otimes A_j)/I)$ for some ideal I. $\square$

Corollary 6.12. *$\mathbb{P}^n$ is separated.*

Proof. Clearly, the standard cover we used to define $\mathbb{P}^n$ satisfies the condition of Proposition 6.11. $\square$

We now are ready to define varieties:

Definition 6.13. *Let k be an algebraically closed field. A* variety *is a scheme over k which is reduced, irreducible, separated and of finite type.*

Irreducible here means that it is irreducible as a topological space, that is the underlying space cannot be written as the union of two closed subspaces, neither of which is the whole space.

Remark 6.14. Note that every irreducible closed subspace of a variety is again a variety equipped with the reduced scheme structure. The only slightly subtle aspect might be to check that it is of finite type, but this is a condition that can be checked locally where it is clear (since it is a quotient of a finitely generated algebra). Also the fact that it is again separated follows from the next exercise sheet (Exercise 2). Also note that by Hilbert's basis theorem (Proposition 4.16) every variety is noetherian. Thus if we drop the condition of irreducibility, i.e. consider X a reduced scheme of finite type over k then we would be able to write it as a finite union of irreducible components (according to Proposition 5.13), each of which is a variety. We will shed light on the irreducibility condition soon.

Example 6.15. $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are both varieties.

Proposition 6.16. *The category of affine varieties over* $\operatorname{Spec}(k)$ *is equivalent to the category of irreducible affine algebraic sets over* k.

Proof. By Theorem 4.10 from last semester, the construction which takes an affine algebraic set $V \subseteq k^n$ to its algebra of functions $\mathcal{O}(V)$ defines an equivalence between the categories

$$\mathrm{AffVar}_k^{\mathrm{op}} \to \mathrm{Alg}_k^{\mathrm{f.g.,red.}}.$$

We can also view this as a covariant equivalence between AffVar_k and the full subcategory of schemes over $\operatorname{Spec}(k)$ which are affine, reduced and of finite type. So we also get an equivalence between the full subcategories of irreducibles on both sides. □

To get a better feeling for varieties (and why we included irreducibility), we introduce the following notion:

Definition 6.17. *A nonempty scheme X is* integral *(as in integral domain, not as in integral extension) if $\mathcal{O}_X(U)$ is an integral domain for all affine opens $U \neq \emptyset$ in X.*

Remark 6.18. This is not a local condition! Indeed, $\operatorname{Spec}(k) \amalg \operatorname{Spec}(k) = \operatorname{Spec}(k \times k)$ does not have global sections given by an integral domain, but of course admits a cover by two copies of $\operatorname{Spec}(k)$, where k is an integral domain.

Proposition 6.19. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. Assume first that X is integral. Then $\mathcal{O}_X(U)$ is an integral domain for each affine U, in particular $\mathcal{O}_X(U)$ is recuced. Using Proposition 13.5 we deduced that X is reduced. If X was not irreducible, we would find nonempty opens U, V with $U \cap V = \emptyset$. By shrinking them we may assume they are affine. But then $\mathcal{O}_X(U \cup V) \cong \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf condition, which is not a domain.

Conversely, assume X is reduced and irreducible. For affine U, given $f, g \in \mathcal{O}_X(U)$ with $f \cdot g = 0$, we need to show that $f = 0$ or $g = 0$. We get closed subspaces $V(f), V(g) \subseteq U$ where f and g vanish. We have that $\overline{U} = X$ by irreducibility. By assumption, $V(f) \cup V(g)$ cover all of U, so $\overline{V(f)} \cup \overline{V(g)}$ cover all of X. By irreducibility, one of them agrees with X, say $\overline{V(f)}$. Then $V(f) = \overline{V(f)} \cap U = U$. It follows that f vanishes at all points of U. Since X is reduced, this means that f is actually zero in $\mathcal{O}_X(U)$. □

Thus we can equally well characterize varieties as integral, separated schemes of finite type. Integral schemes play an important role, since we can study the rings $\mathcal{O}_X(U)$ for U affine by embedding them into their fields of fractions which are subject to the following statement. Recall that the topological space underlying a scheme

is sober, that is each irreducible closed subset has a unique generic point. Strictly speaking we have only shown this for affine schemes, but for a general scheme X we simply note that if $Z \subseteq X$ is irreducible closed then for any affine U we have that $Z \cap U \subseteq U$ is also irreducible closed, thus of the form $\overline{x}$ for a generic point x. Then $Z = \overline{U \cap Z} = \overline{x}$.
In particular every irreducible scheme X has a unique generic point x.

PROPOSITION 6.20. *For any integral scheme (in particular a variety) all the fraction fields of $\mathcal{O}_X(U)$ for U affine open are isomorphic to one another and to the stalk $\mathcal{O}_{X,x}$ at the generic point x of X, which is also isomorphic to the fraction field $\kappa(x)$.*

PROOF. Let $U = \mathrm{Spec}(R)$ be any affine open. Then $x \in U$ since it is a generic point and R is an integral domain. The generic point is the zero ideal in R thus the stalk at the point x agrees with the fraction field of R, in particular is a field. $\square$

For a variety X over k, the stalk $\mathcal{O}_{X,x}$ is a field extension of k, called *field of functions* of X, which remembers some (but not all) information on X. For example, both $\mathbb{A}^1$ and $\mathbb{P}^1$ both have the same field of functions $k(x)$. If X is affine, i.e. of the form $\mathrm{Spec}(A)$ for A a finitely generated k-algebra without zero divisors, then the field of functions is the fraction field of A.

7. Projective varieties

We have seen that irreducible affine algebraic sets over an algebraically closed field k (or equivalently: $\mathrm{Spec}(A)$ for finitely generated reduced integral domains A over k) give examples of varieties. We will now study interesting non-affine examples, based on projective space $\mathbb{P}^n_k$.

DEFINITION 7.1. *A scheme X over a field k is called* projective *if there exists a closed immersion $X \to \mathbb{P}^n_k$ for some n.*

Note that we do not required projective schemes to be reduced, so they are not necessarily varieties.

REMARK 7.2. Note that such projective scheme X is automatically separated and of finite type. The definition of projective schemes is closely related to the following characterisation of affine schemes of finite type over k: X (of finite type over k) is affine if and only if there exists a closed immersion into $\mathbb{A}^n$ for some n. Indeed, closed subschemes of $\mathbb{A}^n$ are affine and of finite type, and vice versa, if $X = \mathrm{Spec}(A)$ with A finitely generated, there exists a surjective map $k[x_1, \ldots, x_n] \to A$.

Recall that the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$ has global sections given by the homogeneous coordinates x_i. Thus, any homogeneous polynomial f of degree d in $x_0, \ldots, x_n$ defines a section $f \in \mathcal{O}(d)(\mathbb{P}^n)$. To a section of a line bundle, we previously assigned an open subscheme $D(f)$ where f was invertible. We now want to see that we can also assign closed subschemes (where f vanishes), and more systematically see that all closed subschemes of $\mathbb{P}^n$ arise as vanishing loci of homogeneous polynomials in this way.

PROPOSITION 7.3. *For a scheme X, closed subschemes $Z \subseteq X$ are in one-to-one correspondence with quasicoherent ideal subsheaves in $\mathcal{O}_X$, i.e. quasicoherent subsheaves*

$$\mathcal{I} \subseteq \mathcal{O}_X$$

such that locally, $\mathcal{I}(U) \subseteq \mathcal{O}_X(U)$ is an ideal. For a scheme Y, a morphism $g : Y \to X$ factors through $Z \subseteq X$ precisely if the morphism $g^\sharp : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ vanishes on $\mathcal{I}$ or equivalently

$$g^*(\mathcal{I}) \to g^*(\mathcal{O}_X) \cong \mathcal{O}_Z$$

is the zero map.

Proof. For a closed subscheme $i : Z \hookrightarrow X$, we define an ideal subsheaf $\mathcal{I}$ as kernel of the structure map $\mathcal{O}_X \to i_*\mathcal{O}_Z$. Recall that there exists at most one isomorphism between two closed subschemes over X, so if Z, Z' are isomorphic locally (over some cover of X), the isomorphisms glue to a well-defined isomorphism over X. So two closed subschemes are isomorphic over X if and only if their ideal sheaves are equal as subsheaves of $\mathcal{O}_X$: Affine-locally, this is just the isomorphism theorem of rings. To see that every ideal subsheaf actually defines a closed subscheme, it again suffices to know this affine-locally (where it is clear), and then glue the resulting schemes along the unique isomorphisms over X.
The description of morphisms then also follows from the affine case. □

Note that the description of morphisms as above in particular gives a description of the functor of points of $Z \subseteq X$ for a given $\mathcal{I}$: the set $Z(R)$ is the subset of $X(R)$ consisting of those elements $g : \mathrm{Spec}(R) \to X$ which satisfy the given description.

Definition 7.4. *Let $\mathcal{E}$ be a vector bundle on a scheme. Then the dual vector bundle $\mathcal{E}^\vee$ is the quasicoherent sheaf with*

$$\mathcal{E}^\vee(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \mathcal{O}_U).$$

This is a sheaf of $\mathcal{O}_X$-modules with the obvious restriction map and the $\mathcal{O}_X$-action given by multiplication. This construction has already been used in the proof of Theorem 2.25. We claim that $\mathcal{E}^\vee$ is again a vector bundle. Since this condition can be checked locally we can assume that $\mathcal{E}|_U = \mathcal{O}_U^n$ and then we get that $\mathcal{E}^\vee|_U \cong \mathcal{O}_U^n$ as well.

Example 7.5. We have that $\mathcal{O}_X^\vee = \mathcal{O}_X$ and for a line bundle $\mathcal{L}$ we have that $\mathcal{L}^\vee$ is the inverse $\mathcal{L}^{-1}$. This is clear locally and then follows globally (remember that the inverse was well-defined).

Note that we have that $(\mathcal{E}^\vee)^\vee$ is canonically isomorphic to $\mathcal{E}$ by the map

$$\mathcal{E} \to (\mathcal{E}^\vee)^\vee \qquad s \in \mathcal{E}(U) \mapsto \big(\mathrm{ev}_s : \mathcal{E}^\vee|_U \to \mathcal{O}_U\big)$$

where ev_s sends $f \in \mathcal{E}^\vee(V) = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{E}|_V, \mathcal{O}_V)$ for $V \subseteq U$ to $f(s|_V)$. Locally on modules this is simply the canonical map from a projective module P to its bidual. For a morphism $f : \mathcal{E} \to \mathcal{F}$ between vector bundles (by which we mean an $\mathcal{O}_X$-linear map) we have a dual map

$$f^\vee : \mathcal{F}^\vee \to \mathcal{E}^\vee$$

locally given by composition with f. We again have that $(f^\vee)^\vee = f$ under the identifications above. Said differently: if we denote by $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)$ the full subcategory of vector bundles then we have the functor

$$(-)^\vee : \mathrm{Vect}(X)^{\mathrm{op}} \to \mathrm{Vect}(X) \qquad \mathcal{E} \mapsto \mathcal{E}^\vee$$

which comes with a natural equivalence $\mathrm{id} \simeq ((-)^\vee)^\vee$. Here we have to consider one of the the instances of $(-)^\vee$ as a functor $\mathrm{Vect}(X) \to \mathrm{Vect}(X)^{\mathrm{op}}$. We see that this shows that $(-)^\vee$ is an equivalence of categories.

DEFINITION 7.6. *For a scheme X, a vector bundle $\mathcal{E}$, and a section $s : \mathcal{O}_X \to \mathcal{E}$, we define a closed subscheme $V(s)$ of X as closed subscheme corresponding to the ideal sheaf $\mathcal{I}$, obtained as image of the dual map*

$$\mathcal{E}^\vee \to \mathcal{O}_X^\vee \cong \mathcal{O}_X.$$

More generally, for $s_1, \ldots, s_n$ sections of different vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$, we write $V(s_1, \ldots, s_n)$ for $V(s)$ where

$$s : \mathcal{O}_X \to \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_n$$

is the combined map. Of course this is just a special case of the first by taking the direct sum of vector bundles.

On every affine open $U = \mathrm{Spec}(A) \subseteq X$, the section s corresponds to an A-linear map $A \to A^n$, i.e. to a list of elements $f_1, \ldots, f_n \in A$. The dual map is given by

$$A^n \to A, \quad (a_1, \ldots, a_n) \mapsto \sum a_i f_i,$$

whose image is precisely the ideal $(f_1, \ldots, f_n)$. So on each affine open, $V(s)$ is of the form $A/(f_1, \ldots, f_n)$, where the f_i are the coordinates of s in a local trivialisation of $\mathcal{E}$. The construction simply tells us a priori that this glues and is independent of the trivialisation.
Note that according to Proposition 7.3 the functor of points of $V(s)$ is given by the subset of $X(R)$ consisting of those morphisms $g : \mathrm{Spec}(R) \to X$ such that the composition

$$\mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_X \xrightarrow{g^\sharp} g_*(\mathcal{O}_{\mathrm{Spec}(R)})$$

is zero. Equivalently the adjoint morphism

$$g^*(\mathcal{E}^\vee) = g^*(\mathcal{E})^\vee \xrightarrow{g^*(s)^\vee} \mathcal{O}_{\mathrm{Spec}(R)}$$

is zero.

PROPOSITION 7.7. *For every commutative ring k we have that*

$$\mathcal{O}(d)(\mathbb{P}^n_k) = \begin{cases} \text{homogeneous polynomials over } k \text{ in } x_0, \ldots, x_n \text{ of degree } d \text{ if } d \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

PROOF. Exercise 3 on the current sheet. □

Here the x_i are the standard sections of $\mathcal{O}(1)(\mathbb{P}^n)$ constructed in Example 2.18. Thus this proposition really says that a specific map

$$k[X_0, ..., X_n]_d \to \mathcal{O}(d)(\mathbb{P}^n_k) \qquad f \mapsto s_f$$

is an isomorphism. Now we can form $V(s_f)$ as in Definition 7.6.
Recall the functor of points description of $\mathbb{P}^n_k$ given as

$$\mathbb{P}^n_k(A) = \{\mathcal{L} \subseteq A^{n+1} \mid \mathcal{L} \text{ locally complementable and invertible}\}$$

for any k-algebra A (here this is the relative functor of points).

PROPOSITION 7.8. *For any homogenous polynomial $f \in k[X_0, ..., X_n]_d$ we consider the associated polynomial map $\overline{f}_A : A^{n+1} \to A$. Then the closed subscheme $V(s_f) \subseteq \mathbb{P}^n_k$ has the following concrete functor of points description:*

$$V(s_f)(A) = \{\mathcal{L} \in \mathbb{P}^n_k(A) \mid \overline{f}_A(l) = 0 \quad \forall l \in \mathcal{L}\}\ .$$

EXAMPLE 7.9. Assume that $A = k$. Then $\mathcal{L} = [a_0 : ... : a_n]$ and we have that $\overline{f}_k$ vanishes on that line, precisely if $f(a_0, ..., a_n) = 0$, which is a well-defined condition since f is homogenous.

Note that for a map of k-algebras $A \to B$ we have the induced map

$$\mathbb{P}^n_k(A) \to \mathbb{P}^n_k(B)$$

and it takes the subset $V(s_f)(A)$ described in the Proposition to the subset $V(s_f)(B)$ so that we indeed describe a functor this way.

PROOF OF PROPOSITION 7.8 . Let us first make the section s_f explicit. Concretely for a homogenous polynomial $f \in k[x_0, ..., x_n]_d$ we have to describe a morphism

$$\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(d)$$

of line bundles on $\mathbb{P}^n$. Dually this is a morphism

$$\mathcal{O}(-d) \to \mathcal{O}_{\mathbb{P}^n} \ .$$

Here $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$. On $\mathbb{P}^n_k$ we have a canonical inclusion of vector bundles

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n}$$

After pullback along $\mathrm{Spec}(A) \to \mathbb{P}^n_k$ this map is given by the inclusion $\mathcal{L} \subseteq A^{n+1}$. The x_i's are simply the composition with the projections to the coordinates of $\mathcal{O}^{n+1}_{\mathbb{P}^n}$. Then the map associated to a homogenous polynomial $f = x_{i_1} \cdot ... \cdot x_{i_d}$ is given by the sum of d-fold tensor products of these maps. In particular for any morphism $g : \mathrm{Spec}(A) \to \mathbb{P}^n$ corresponding to $\mathcal{L} \subseteq A^{n+1}$ we have that the corresponding morphism

$$g^*(s_f^\vee) : g^*\mathcal{O}(-d) \to g^*\mathcal{O}_{\mathbb{P}^n}$$

which is a map

$$\tilde{f}_A : \mathcal{L}^{\otimes d} \to A \ .$$

Concretely this map sends $l_1 \otimes ... \otimes l_d$ to $(l_1)_{i_1} \cdot ... \cdot (l_d)_{i_d}$. In particular we see for a monomial f and an element $l \in L$ that we have

$$\tilde{f}_A(l \otimes ... \otimes l) = \overline{f}_A(l) \ . \tag{13}$$

The equality (13) then also holds for arbitrary homogenous polynomials by linear extension.

In particular for a given $\mathcal{L} \subseteq A^{n+1}$ we have that

$$\overline{f}_A(l) = 0 \forall l \Leftrightarrow \tilde{f}_A(l \otimes ... \otimes l) = 0 \forall l \ .$$

In order to prove the Proposition we need to verify therefore that $\tilde{f}_A(l \otimes ... \otimes l) = 0$ for all $l \in \mathcal{L}$ already implies that the map

$$\tilde{f}_A : \mathcal{L}^{\otimes d} \to A$$

vanishes. Since line bundles are locally trivial we can pick elements $f_i \in A$ and non-vanishing sections s_i of $\mathcal{L}|_{D(f_i)}$, i.e. $\mathcal{L}|_{D(f_i)} = R[f_i^{-1}] \cdot s_i$. We can replace s_i by $s_i \cdot f_i^n$ for n sufficiently large and thus assume that s_i is a section of $\mathcal{L}$ which after localization at f_i generates $\mathcal{L}$. Then we have that

$$\tilde{f}_A(s_i \otimes ... \otimes s_i) = 0 \ .$$

Now to check that

$$\tilde{f}_A : \mathcal{L}^{\otimes d} \to A$$

is zero, we can check this locally. But locally $s_i \otimes ... \otimes s_i$ generated $\mathcal{L}^{\otimes d}$ so that we are done. □

THEOREM 7.10. *Let $X \subseteq \mathbb{P}^n_k$ for k a field be a closed subscheme* [6]*. Then $X = V(f_1, \dots, f_k)$ for homogeneous polynomials f_i of degree d (we can take the same d for all i).*

We will prove this result eventually, but first we want to develop some terminology.

DEFINITION 7.11. *Let X be a scheme. We say that a line bundle $\mathcal{L}$ on X is ample if X is quasi-compact and for each $x \in X$ there exists $m \geq 1$ and a section $s \in \mathcal{L}^{\otimes m}(X)$ such that*

$$D(s) \subseteq X$$

is an affine open neighborhood of x.

EXAMPLE 7.12.

(1) On an affine scheme Spec(R), the trivial line bundle $\mathcal{O}_{\mathrm{Spec}(R)}$ is ample.
(2) The line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n_k$ is ample. Indeed, it has the sections x_i with $D(x_i)$ the standard cover of $\mathbb{P}^n_k$.

LEMMA 7.13. *$\mathcal{L}$ is ample precisely if there exists a number $m \geq 1$ and sections $s_1, ..., s_n \in \mathcal{L}^{\otimes m}(X)$ such that the $D(s_i)$ cover X.*
A line bundle $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes m}$ is ample for some $m \geq 1$ iff $\mathcal{L}^{\otimes m}$ is ample for all $m \geq 1$.

PROOF. If the condition of the Lemma holds, then clearly $\mathcal{L}$ is ample. Conversely assume that $\mathcal{L}$ is ample. Then we pick m_i and $s'_i \in \mathcal{L}^{\otimes m_i}(X)$ such that $D(s_i)$ cover X. By quasicompactness, we can assume these are finitely many. Then we set

$$m := m_1 \cdot ... \cdot m_n \ .$$

Then we also set $s_i := (s'_i)^{m/m_i} \in \mathcal{L}^{\otimes m}(X)$. We have that $D(s_i) = D(s'_i)$ and therefore the condition of the Lemma.
Finally if $\mathcal{L}$ is ample, take sections $s_1, \dots, s_n \in \mathcal{L}^{\otimes m}(X)$ such that the $D(s_i)$ cover X. Then also $s_1^k, \dots, s_n^k \in \mathcal{L}^{\otimes mk}(X)$ have this property, so $\mathcal{L}^{\otimes k}$ is ample. The converse is clear. □

EXAMPLE 7.14. The line bundles $\mathcal{O}(d)$ on $\mathbb{P}^n_k$ are ample for $d \geq 1$ but not for $d \leq 0$. In the latter case no power of $\mathcal{O}(d)$ has any non-constant global sections.

LEMMA 7.15. (1) *On an affine scheme* Spec(R)*, any line bundle is ample.*
(2) *If $\mathcal{L}$ is a line bundle on* Spec(R) *and $s \in \mathcal{L}(\mathrm{Spec}(R))$, the open subset $D(s) \subseteq \mathrm{Spec}(R)$ is affine.*

PROOF. We first prove the second part. Write $L = \mathcal{L}(\mathrm{Spec}(R))$, view the section s as map $R \to L$, and define

$$R[\frac{1}{s}] := \mathrm{colim}\left(R \xrightarrow{s} L \xrightarrow{s} L^{\otimes 2} \xrightarrow{s} \dots\right).$$

Here, elements should be thought of as formal fractions $\frac{a}{s^k}$ with $a \in L^{\otimes k}$. We claim that Spec$(R[\frac{1}{s}]) \to$ Spec(R) is an open immersion with image $D(s)$. If L is the trivial line bundle, where s is precisely an element of R, this is just the usual $D(s)$

[6]Actually k being noetherian is enough

and the usual $R[\frac{1}{s}]$. More generally, we can check the claim locally, but locally $\mathcal{L}$ is trivial.
To check the first statement, observe that for each $x \in \operatorname{Spec}(R)$, we find a neighbourhood $D(f)$ on which $\mathcal{L}$ is trivial, i.e. $\mathcal{L}(D(f)) = L[\frac{1}{f}] \cong R[\frac{1}{f}]$. Denote the element corresponding to 1 by $\frac{s}{f^k}$, then $D(f) \subseteq D(s)$. By the second statement, $D(s)$ is indeed an affine neighbourhood. $\square$

LEMMA 7.16. *If X is a scheme with an ample line bundle $\mathcal{L}$, X is quasicompact and quasiseparated.*

PROOF. Quasicompact is part of the definition of ample line bundles. For quasiseparatedness, first observe that if $D(s)$, $D(s')$ are affine opens for $s, s' \in \mathcal{L}^{\otimes m}(X)$, then

$$D(s) \cap D(s') \subseteq D(s)$$

is the open subset associated to $s' \in \mathcal{L}^{\otimes m}(D(s))$. By the previous lemma, this is automatically affine, in particular quasicompact. So X has a cover by affines, the intersections of which are affine, in particular quasicompact. This implies that X is quasiseparated, we can directly argue as below:
let U, V be two quasicompact subsets. Each $x \in U \cap V$ is contained in some $D(s)$. We choose opens $W_x \subseteq U \cap D(s)$ and $W'_x \subseteq V \cap D(s)$ which are principal opens in $D(s)$. Finitely many of the W_x cover U, denote those by $W_1, \dots, W_n$, and analogously $W'_1, \dots, W'_m$ cover V. Each $W_i \cap W'_j$ is a principal open in $D(s) \cap D(s')$. So $U \cap V$ is the union of finitely many quasicompacts, therefore it is quasicompact. $\square$

One of the main benefits of ample line bundles is that they generally allow us to describe sections on opens as a kind of localisation:

LEMMA 7.17. *If X is quasicompact and quasiseparated, $\mathcal{L}$ is a line bundle on X, $s \in \mathcal{L}(X)$, and $\mathcal{M}$ is a quasicoherent sheaf on X, then local sections $\mathcal{M}(D(s))$ admit the description*

$$\mathcal{M}(D(s)) = \operatorname{colim}\Big(\mathcal{M}(X) \xrightarrow{s} (\mathcal{M} \otimes \mathcal{L})(X) \xrightarrow{s} (\mathcal{M} \otimes \mathcal{L}^{\otimes 2})(X) \xrightarrow{s} \dots\Big)$$

PROOF. If X is affine and $\mathcal{L}$ is trivial, this is simply the formula

$$\mathcal{M}(D(s)) = \mathcal{M}(X)[\frac{1}{s}].$$

In general, cover X by finitely many affine opens U_i on which $\mathcal{L}$ is trivial, and cover each intersection U_{ij} by finitely many affine opens U_{ijk}. The sheaf condition describes

$$(\mathcal{M} \otimes \mathcal{L}^{\otimes k})(X) \cong \operatorname{eq}\big(\textstyle\bigoplus_{U_i} (\mathcal{M} \otimes \mathcal{L}^{\otimes k})(U_i) \rightrightarrows \bigoplus_{U_{ijk}} (\mathcal{M} \otimes \mathcal{L}^{\otimes k})(U_{ijk})\big)$$

and

$$\mathcal{M}(D(s)) \cong \operatorname{eq}\big(\textstyle\bigoplus_{U_i} \mathcal{M}(U_i \cap D(s)) \rightrightarrows \bigoplus_{U_{ijk}} \mathcal{M}(U_{ijk} \cap D(s))\big).$$

Since filtered colimits commute with equalizers and direct sums (here we used qcqs, otherwise we would have infinite products here), and we have already checked the claim on the U_i, passing to the colimit yields the claim. $\square$

Note that $\mathcal{M}|_{D(s)}$ and $(\mathcal{M} \otimes \mathcal{L}^{\otimes k})|_{D(s)}$ are isomorphic via the multiplication by s^k map. So in particular, the above lemma says that any local section in $\mathcal{M}(D(s))$ comes from a global section of some $\mathcal{M} \otimes \mathcal{L}^{\otimes k}$ after multiplying with some s^k. That is we can think of elements in $\mathcal{M}(D(s))$ as fractions $\frac{g}{s^k}$ for $g \in \mathcal{M}(X)$. This is a strong generalisation of the usual observation in the affine situation, that every element of $\mathcal{M}(D(f)) = M[\frac{1}{f}]$ can be viewed as $\frac{m}{f^k}$ for an element $m \in M$.
We make a number of observations:

Corollary 7.18. *If $\mathcal{L}$ is a line bundle on a quasicompact and quasiseparated X, and $D(s)$ for $s \in \mathcal{L}(X)$ is affine, its ring of functions is explicitly given by*

$$\operatorname{colim}\left(\mathcal{O}_X(X) \xrightarrow{s} \mathcal{L}(X) \xrightarrow{s} \mathcal{L}^{\otimes 2}(X) \xrightarrow{s} \dots\right).$$

Explicitly, elements can be thought of as formal fractions $\frac{a}{s^k}$ with $a \in \mathcal{L}^{\otimes k}(X)$.

Proof. If $D(s)$ is affine, its ring of functions is given by $\mathcal{O}_X(D(s))$. This has the claimed formula by the lemma. $\square$

Corollary 7.19. *If $\mathcal{L}$ is an ample line bundle on X, X is already separated.*

Proof. By replacing $\mathcal{L}$ by a suitable power, we may assume that X has a cover by affines $D(s_i)$ with $s_i \in \mathcal{L}(X)$. We claim that $D(s_i) \cap D(s_j)$ is also affine, and the map

$$\mathcal{O}_X(D(s_i)) \otimes \mathcal{O}_X(D(s_j)) \to \mathcal{O}_X(D(s_i) \cap D(s_j))$$

is surjective. In fact, $D(s_i) \cap D(s_j)$ is simply the principal open subset of $D(s_i)$ obtained by inverting $\frac{s_j}{s_i}$. In particular, every element of $\mathcal{O}_X(D(s_i) \cap D(s_j))$ is hit by some

$$\frac{a}{s_i^k} \otimes \frac{s_i^l}{s_j^l}.$$

which shows surjectivity. $\square$

Corollary 7.20. *A line bundle $\mathcal{L}$ on a quasicompact scheme X is ample if and only if the opens $D(s)$ for $s \in \mathcal{L}^{\otimes m}(X)$ for all m form a basis of the topology of X.*

Proof. If $\mathcal{L}$ is ample, take $x \in U$. We find an open affine $x \in D(s) = \operatorname{Spec}(R)$ with $s \in \mathcal{L}^m(X)$, and take a principal open $x \in D(f) \subseteq D(s) \cap U$ for $f \in R$. Use Corollary 7.18 to write $f = \frac{a}{s^k}$ with $a \in \mathcal{L}^{\otimes mk}(X)$, then $D(as) = D(f)$ and $as \in \mathcal{L}^{\otimes m(k+1)}(X)$.
Conversely, if the $D(s)$ for $s \in \mathcal{L}^{\otimes m}(X)$ form a basis of the topology of X, we need to check that the *affine* such $D(s)$ still cover X. Take some $x \in X$ and an affine open neighbourhood U. Then there exists $D(s) \subseteq U$. By Lemma 7.15, this is automatically affine. $\square$

Corollary 7.21. *Let $\mathcal{M}$ be a quasicoherent sheaf of finite type on a scheme X, and let $\mathcal{L}$ be an ample line bundle. Then there exists m and a finite list of sections $f_1, \dots, f_n \in (\mathcal{M} \otimes \mathcal{L}^{\otimes m})(X)$ which generate $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$, in the sense that*

$$\bigoplus \mathcal{O}_X \xrightarrow{f_1, \dots, f_n} \mathcal{M} \otimes \mathcal{L}^{\otimes m}$$

is an epimorphism of sheaves.

Proof. Cover X by finitely many affine opens $D(s_i)$ where $s_i \in \mathcal{L}(X)$. On each $D(s_i)$, $\mathcal{M}$ restricts to a finitely generated module over $\mathcal{O}_X(D(s_i))$, so there is a finite list of sections $f_{ij} \in \mathcal{M}(D(s_i))$ which generate, i.e. that

$$\bigoplus_j \mathcal{O}_X|_{D(s_i)} \to \mathcal{M}|_{D(s_i)}$$

is an epimorphism. Multiplying with s_i^m for m large enough, we may assume that each $f'_{ij} := s_i^m f_{ij}$ comes from a global section of $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$, by Lemma 7.17. So the map

$$\bigoplus_j \mathcal{O}_X \xrightarrow{f'_{ij}} \mathcal{M} \otimes \mathcal{L}^{\otimes m}$$

is an epimorphism on $D(s_i)$, and the sum of all,

$$\bigoplus_{i,j} \mathcal{O}_X \xrightarrow{f'_{ij}} \mathcal{M} \otimes \mathcal{L}^{\otimes m}$$

is an epimorphism globally, as desired. □

We are now ready to prove Theorem 7.10.

Proof of Theorem 7.10. Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme. We have a corresponding ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_k}$. It is finite type since $\mathbb{P}^n_k$ is Noetherian[7]. Since $\mathcal{O}(1)$ is ample, $\mathcal{I} \otimes \mathcal{O}(d)$ is globally generated for some d, i.e. we find an epimorphism

$$\bigoplus \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{I} \otimes \mathcal{O}(d),$$

or equivalently an epimorphism

$$\bigoplus \mathcal{O}(-d) \to \mathcal{I}.$$

This characterizes $\mathcal{I}$ as the image of the composite $\bigoplus \mathcal{O}(-d) \to \mathcal{O}$. If we write $f_1, \dots, f_n$ for the sections of $\mathcal{O}(d)$ described by the dual map $\mathcal{O} \to \bigoplus \mathcal{O}(d)$, we have thus identified

$$X = V(f_1, \dots, f_n),$$

by definition of the latter. □

Having convinced ourselves of the usefulness of ample line bundles, we might wonder: Which schemes do admit ample line bundles?

Lemma 7.22. *If $i : X \to Y$ is a locally closed immersion and $\mathcal{L}$ is an ample line bundle on Y, then $i^*\mathcal{L}$ is ample on X.*

Proof. We prove this separately for open and closed immersions. For open immersions, this is a direct consequence of Corollary 7.20 since for $U \subseteq X$ and $s \in \mathcal{L}(X)$ we have $D(s|_U) = D(s) \cap U$.
So let $i : X \to Y$ be a closed immersion, and let $D(s_i)$ be an affine cover of Y with $s_i \in \mathcal{L}^{\otimes m}(Y)$. The s_i pull back to global sections $t_i \in (i^*\mathcal{L})^{\otimes m}(X)$, and we have $D(t_i) = i^{-1}(D(s_i))$. Finally, $i : D(t_i) \to D(s_i)$ is a closed immersion, so $D(t_i)$ is affine. □

As our prototypical example of an ample line bundle is on $\mathbb{P}^n_k$, we make the following definition:

[7] Here is where the assumption that k is noetherian enters and it can obviously be replace by k a noetherian ring or even scheme

Definition 7.23. *A scheme X over* $\mathrm{Spec}(k)$ *is quasi-projective if there exists a locally closed immersion*

$$X \to \mathbb{P}^n_k$$

for some n.

By the above, every quasiprojective scheme admits an ample line bundle. This includes projective schemes, but also affine schemes of finite type, since we have open immersions $\mathbb{A}^n_k \to \mathbb{P}^n_k$. Surprisingly, the converse is true as well:

Theorem 7.24. *A scheme X of finite type over* $\mathrm{Spec}(k)$ *admits an ample line bundle if and only if is quasi-projective.*

This clarifies the important role that projective space plays in algebraic geometry. Note that many schemes come with an ample line bundle but no obvious immersion into $\mathbb{P}^n_k$, for example products $\mathbb{P}^n_k \times \mathbb{P}^m_k$.

Proof of Theorem 7.24. If X is quasi-projective, then it admits an ample line bundle: Simply pull back $\mathcal{O}_{\mathbb{P}^n_k}(1)$ along a locally closed immersion $i : X \to \mathbb{P}^n_k$ and apply Lemma 7.22.
Conversely, assume X has an ample line bundle $\mathcal{L}$, we show that it is quasi-projective. By replacing $\mathcal{L}$ by powers if necessary, we may assume that X can be covered by finitely many affine opens $D(s_i)$, $0 \leq i \leq n$, for $s_i \in \mathcal{L}(X)$. Since X is of finite type, each

$$\mathcal{O}_X(D(s_i)) = \mathrm{colim}\left(\mathcal{O}_X(X) \xrightarrow{s_i} \mathcal{L}(X) \xrightarrow{s_i} \mathcal{L}^{\otimes 2}(X) \xrightarrow{s_i} \dots\right)$$

is a finitely generated k-algebra. Represent generators by elements of the form $\frac{a_{ij}}{s_i^m}$ for a common large enough m, with $a_{ij} \in \mathcal{L}^{\otimes m}(X)$.
The sections $s_0^m, \dots, s_n^m, (a_{ij})_{i,j}$ define a map

$$\mathcal{O}_X^{n+1+N} \to \mathcal{L}^{\otimes m}.$$

This is an epimorphism, since on $D(s_i)$, s_i^m is even an isomorphism. This also shows that the map splits locally: On $D(s_i)$, a right inverse is given by the map $s_i^{-m} : \mathcal{L}^{\otimes m} \to \mathcal{O}_X$ into the i-th summand. Dually,

$$\mathcal{L}^{-m} \to \mathcal{O}_X^{n+1+N}$$

is locally complementable. Thus this datum defines a map $f : X \to \mathbb{P}^{n+N}_k$ by Theorem 2.15.
We claim it is a locally closed immersion. By construction, f^* takes

$$\left(\mathcal{O}(-1) \to \mathcal{O}^{n+1+N}_{\mathbb{P}^{n+N}_k}\right) \mapsto \left(\mathcal{L}^{-m} \to \mathcal{O}_X^{n+1+N}\right)$$

and dually, since f^* preserves duality as we will see in the exercises:

$$\left(\mathcal{O}^{n+1+N}_{\mathbb{P}^{n+N}_k} \to \mathcal{O}(1)\right) \mapsto \left(\mathcal{O}_X^{n+1+N} \to \mathcal{L}^{\otimes m}\right).$$

The map on the left is given by the homogeneous coordinates $x_0, \dots, x_{n+N}$, so under f, they pull back to $s_0^m, \dots, s_n^m, (a_{ij})_{i,j}$. In particular, the preimage of U_i under f is given by $f^{-1}(D(x_i)) = D(s_i)$. Since the $D(s_i)$ cover X it remains to check that $D(s_i) \to D(x_i)$ is a closed immersion. But by construction, the $\frac{a_{ij}}{s_i^m}$ generate $\mathcal{O}_X(D(s_i))$, so the map

$$\mathcal{O}_{\mathbb{P}^{n+N}}(U_i) \to \mathcal{O}_X(D(s_i))$$

is surjective. Since both are affine, this proves the claim. $\square$

REMARK 7.25. The existence of ample line bundles gives us thus a sufficient and necessary condition for being quasi-projective (under the additional finite type assumption which is also necessary). One can wonder if one can give a similar description for projective schemes. And this is indeed possible: being a closed subscheme of $\mathbb{P}^n_k$ one can deduce that projective schemes satisfy the algebro-geometric analogue of being compact. This is called proper. We will not get into the intricacies of proper scheme here, but the final result is that all projective schemes are proper and conversely a scheme over k is projective if and only if it is of finite type, proper and admits an ample line bunde.

8. Line bundles and the Picard group

In this section, we attempt to answer the question: How many line bundles, up to isomorphism, are there on a given scheme X?

DEFINITION 8.1. *For a scheme X, we define the* Picard group $\mathrm{Pic}(X)$ *to be the group of isomorphism classes of line bundles on X. The group structure is induced by the tensor product of line bundles.*

Note that this is indeed a group: the tensor product of line bundles is again a line bundle and it is associative. The neutral element is the line bundle $\mathcal{O}_X$ and there are inverses given by the inverse bundle (which is also the dual).

REMARK 8.2. There is one slight subtlety though: a priori it is not clear that $\mathrm{Pic}(X)$ is even a set, it could be too big as the collection of all $\mathcal{O}_X$-modules is certainly not a set (neither is the collection of all quasi-coherent sheaves). Even for a field k the collection of all k-vector spaces doesn't form a set, since for each set S we can form the free vector space on that set and two such are isomorphic precisely if the sets are in bijection. Thus the collection of isomorphism classes of all vector spaces contains the collection of all isomorphism classes of all sets which is too big to be a set (by the usual argument).
However, we claim that the Picard group is still a set. This follows from the fact that each line bundle is locally trivial and thus for every line bundle $\mathcal{L}$ we find a cover U_i such that $L|_{U_i}$ is trivializable. But then we can describe L by means of a cocyle, that is a family of isomorphisms

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{O}_{U_i \cap U_j}$$

which satisfy the cocycle identities. The entirety of all such morphisms forms a set. Thus we find that all line bundles that can be trivialized on the given cover form up to isomorphism a set and thus all isomorphism classes of line bundles form a set as the union over all covers of those sets.

Our main tool to study the Picard group is the notion of a divisor.

DEFINITION 8.3. *Let X be a scheme. A closed subscheme $D \subseteq X$ is called effective Cartier divisor if the associated ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_X$ is a line bundle. In this case we denote the dual of $\mathcal{I}_D$ by $\mathcal{O}_X(D)$ and call it the line bundle associated to the divisor.*

Note that the dual of the inclusion $\mathcal{I}_D \to \mathcal{O}_X$ defines a section $1_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ and then we have that $D = V(1_D)$. Thus effective Cartier divisors are in particular vanishing loci of sections of line bundles. Note that the section s is a monomorphism of sheaves (since it is obtained from the inclusion by tensoring with $\mathcal{O}_X(D)$. Conversely if we have a monomorphism of sheaves $s : \mathcal{O} \to \mathcal{L}$ then we get an effective Cartier divisor $V(s) \subseteq X$ with $\mathcal{O}_X(V(s)) = \mathcal{L}$ which follows by tracking through the constructions. We conclude that effective Cartier divisors are in 1-1 correspondence the line bundles equipped with a monomorphic section (up to the 'obvious' choice of isomorphism).

REMARK 8.4. Note that if X is integral (e.g. a variety), the condition that a section $s : \mathcal{O}_X \to \mathcal{L}$ of a line bundle being a monomrophism is equivalent to not being zero. Indeed, if s is nonzero, then also $s|_U : \mathcal{O}_U \to \mathcal{L}|_U$ is nonzero for each nonempty open $U \subseteq X$, since otherwise $V(s) \cup (X \setminus U) = X$, contradicting integrality (Proposition 6.19). Now each point has an affine neighbourhood $U = \mathrm{Spec}(R)$ in which $\mathcal{L}$ is trivial, so in that neighbourhood $s|_U : R \to R$ is simply a nonzero element of R. This is a domain by integrality, so $s|_U : R \to R$ is injective. Thus in this generality effective Cartier divisors are in 1-1 correspondence to line bundles equipped with a non-zero section.

EXAMPLE 8.5. Let $X = \mathrm{Spec}(R)$. Then a closed subscheme $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is an effective Cartier divisor precisely if $I \subseteq R$ is a line bundle, e.g. if I is locally a principal ideal generated by a nn-zero divisor. Then $L = I^\vee$.

EXAMPLE 8.6. The subscheme $\mathbb{P}^n_k \subseteq \mathbb{P}^{n+1}_k$ is an effective Cartier divisor whose corresponding line bundle is $\mathcal{O}(1)$.

Note that the condition of bring a Cartier divisor is a local condition, that is if $D \subseteq X$ is a Cartier divisor if for each $U \subseteq X$ the intersection $U \cap D \to U$ is a Cartier divisor. In particular if D is a point we can check this in a small neighborhood.
The key result to understand the Picard group is now the following:

PROPOSITION 8.7. *Let X be a quasicompact and quasiseparated scheme and $D \subseteq X$ and effecitve Cartier divisor which is also reduced and irreducible.* [8] *Then the sequence*

$$\mathbb{Z} \xrightarrow{\mathcal{O}_X(D)} \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \setminus D)$$

is exact, *meaning that the kernel of the right map agrees with the image of the left map.*

PROOF OF PROPOSITION 8.7. We write $\mathcal{L} = \mathcal{O}_X(D)$ and $s = 1_D$ to simplify notation.
We first note that $\mathcal{L}$ is trivial on $X \setminus D$ where the trivialization is given by the sections s. This shows that the composition of maps is trivial. For the converse we have to show the following: If line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on X become isomorphic after restriction to $D(s)$, they already differ only by a power of $\mathcal{L}$.
Fix an isomorphism $\mathcal{L}_1|_{D(s)} \cong \mathcal{L}_2|_{D(s)}$. This isomorphism and its inverse can be considered as sections of $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$ and $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ on $D(s)$. By Lemma 7.17, we can write both of them as $\frac{f}{s^k}$ and $\frac{g}{s^l}$ for global sections of $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}^{\otimes k}$ and $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \otimes \mathcal{L}^{\otimes l}$. This means we find maps

$$\begin{aligned} f &: \mathcal{L}_1 \to \mathcal{L}_2 \otimes \mathcal{L}^{\otimes k}, \\ g &: \mathcal{L}_2 \to \mathcal{L}_1 \otimes \mathcal{L}^{\otimes l} \end{aligned}$$

[8] This just means it is given by an irreducible closed subset equipped with the reduced subscheme structure.

whose composites are each given by multiplication with s^{k+l}. We now assume we have chosen such maps f, g with k, l arbitrary *integers* with $k+l \geq 0$, in such a way that $k+l$ is as small as possible.[9] If $k+l=0$, we are done: Then the composites of

$$f : \mathcal{L}_1 \to \mathcal{L}_2 \otimes \mathcal{L}^{\otimes k},$$
$$g : \mathcal{L}_2 \otimes \mathcal{L}^k \to \mathcal{L}_1$$

are given by the identities, and so $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathcal{L}^{\otimes k}$, as we wanted to show. So assume $k+l>0$. In that case, we have $V(f) \cup V(g) = V(s)$ as subsets, since f and g compose to a power of s. By irreducibility, one of them agrees, say $V(f) = V(s)$. We claim that then there exists a unique lift in the following diagram:

$$\begin{tikzcd} & \mathcal{L}_2 \otimes \mathcal{L}^{\otimes k-1} \arrow[d, "s"] \\ \mathcal{L}_1 \arrow[ur, dashed] \arrow[r, "f"'] & \mathcal{L}_2 \otimes \mathcal{L}^{\otimes k}. \end{tikzcd}$$

To check this, cover X by affine opens U_i small enough so that all of $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ are trivial on $U_i = \mathrm{Spec}(R_i)$. On each of those opens, our sections trivialize to maps $f|_{U_i} : R_i \to R_i$ and $s|_{U_i} : R_i \to R_i$, i.e. to elements f_i, s_i of R_i, with $V(f_i) = V(s_i)$. So we have

$$\sqrt{(f_i)} = \sqrt{(s_i)} = (s_i),$$

where the latter equality holds since by assumption $V(s_i) = V(s) \cap U_i$ is reduced. So f_i is divisible by s_i, and it follows that we find a lift as desired over each U_i. Since s is a monomorphism of sheaves, it then follows that lifts against s are locally unique, and so the local lifts glue together to a global lift. Denote this lift by f'. Again since s is a monomorphism, it now follows that $f' \circ g$ and $g \circ f'$ are each given by s^{k+l-1}, contradicting the minimality of $k+l$. This finishes the proof. $\square$

EXAMPLE 8.8. Assume that R is a domain. Then a closed subscheme $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ satsifies the assumptions precisely if I is an invertible prime ideal. This is for example the case if $I = (\pi)$ for a prime element $\pi \in R$. In this situation we get from the theorem that

$$\mathrm{Pic}(R) \to \mathrm{Pic}(R[\pi^{-1}])$$

is injective.

COROLLARY 8.9. *If R is a UFD then* $\mathrm{Pic}(\mathrm{Spec}(R))$ *is the trivial group, so every line bundle is trivial on* $\mathrm{Spec}(R)$.

PROOF. Let $\mathcal{L}$ be any line bundle. Since line bundles are locally trivial, $\mathcal{L}$ is trivial on some $D(f)$. Since R is a UFD, we can write $f = f_1 \cdots f_n$ for prime elements $f_i \in R$. Since all of these are sections of the trivial line bundle, Proposition 8.7 shows that all the maps

$$\mathrm{Pic}(\mathrm{Spec}(R)) \to \mathrm{Pic}(D(f_1)) \to \mathrm{Pic}(D(f_1 f_2)) \to \ldots \to \mathrm{Pic}(D(f))$$

are injective (we use that localizations of UFDs are again UFDs). Since $[\mathcal{L}] \mapsto 1$ on the right, $[\mathcal{L}] = 1$ on the left, so $\mathcal{L} \cong \mathcal{O}_{\mathrm{Spec}(R)}$. $\square$

THEOREM 8.10. $\mathrm{Pic}(\mathbb{P}^n_R) \cong \mathbb{Z}$ *for any UFD R, with generator given by $\mathcal{O}(1)$, i.e. any line bundle on $\mathbb{P}^n_R$ for a UFD R is given by $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

[9] Note that we can choose k and l even as natural numbers, but for the next step we would like to be more liberal and allow integers here.

PROOF. Using that $\mathbb{P}^{n-1}_R \subseteq \mathbb{P}^n_R$ is an effective Cartier divisor Proposition 8.7 gives us an exact sequence

$$\mathbb{Z} \xrightarrow{d \mapsto \mathcal{O}(d)} \mathrm{Pic}(\mathbb{P}^n_R) \to \mathrm{Pic}(\mathbb{P}^n_R \setminus \mathbb{P}^{n-1}_R)).$$

Note that $\mathbb{P}^n_R \setminus \mathbb{P}^{n-1}_R \cong \mathbb{A}^n_R \cong \mathrm{Spec}(R[X_{1,0}, \ldots, X_{n,0}])$ is the spectrum of a UFD (by Gauss' Lemma, polynomial rings over UFDs are UFDs). So the right-hand term in our exact sequence is trivial, therefore the map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n_R)$ is surjective. It is also injective: If we had $\mathcal{O}(n) \cong \mathcal{O}(m)$ with $n < m$, then this would mean that $\mathcal{O}(n-m) \cong \mathcal{O}$, but the latter has nonzero sections, the former doesn't since $n - m < 0$. $\square$

We end by a nice application.

PROPOSITION 8.11. *Assume that X is a 1-dimensional, irreducible and noetherian scheme which moreover satisfies the following assumption: every closed point with the reduced subscheme structrue $\{x\} \subseteq X$ is an effective Cartier divisor with associated line bundle $\mathcal{O}(x)$. Then every line bundle on X is isomorphic to a tensor product of line bundles $\mathcal{O}(x_1)^{\otimes k_1} \otimes ... \otimes \mathcal{O}(x_n)^{\otimes k_n}$ for closed points $x_i \in X$ and integers $k_i \in \mathbb{Z}$.*

We will see that the assumption on X is satisfied if X is a smooth curve next term (a notion to be introduced). For the proof we need a lemma:

LEMMA 8.12. *Each quasi-compact scheme X contains a closed point*

PROOF. First we note that affine schemes $X = \mathrm{Spec}(R)$ contain closed points by the existence of maximal ideals. Now for general X pick an irredundant cover by affines

$$X = U_1 \cup ... \cup U_n \ .$$

Then $X \setminus (U_2 \cup ... \cup U_n)$ is non-empty, closed in X and affine. The latter since it is a closed subset of U_1. Thus we find a closed point in this set which is then also closed in X. [10] $\square$

PROOF OF PROPOSITION 8.11. Let $\mathcal{L}$ be an arbitrary line bundle. Then there exists a non-empty open subset $U \subseteq X$ such that $L|_U$ is trivial. Consider the complement Z of U. We claim that U (more generally every open set of X) is of the form $U \supseteq X \setminus \{x_1, ..., x_n\}$ for closed points $x_i \in X$, i.e. $Z = \{x_1, .., x_n\}$: Since X is noetherian so is Z and we can decompose into irreducible components $Z = Z_1 \cup ... \cup Z_i$. Each $Z_i \subseteq X$ contains a point $x_i \in Z$ by the previous lemma. Thus we have the chain of closed irreducible subsets

$$\{x_i\} \subseteq Z_i \subseteq Z$$

and by being 1-dimensional we have that one of the inclusions have to be an equality. If it was the second then U would have to be emtpy, which we excluded. Thus $Z_i = \{x_i\}$ and therefore $Z = \{x_1, ..., x_n\}$ and U has the claimed form.
Now since $\{x_1\}$ is an effective Cartier divisor in X is is also an effective Cartier divisor in the open set $X \setminus \{x_2, ..., x_n\}$ since being an effective Cartier divisor is a local notion. Thus we have an exact sequence

$$\mathbb{Z} \xrightarrow{\mathcal{O}(x_1)} \mathrm{Pic}(X \setminus \{x_2, ..., x_{n-1}\}) \to ... \,\mathrm{Pic}(U \setminus \{x_1, x_2, ..., x_n\})$$

[10]The condition that the scheme is quasi-compact is really necessary to find a non-redundant affine cover. And in fact one can find non-quasi compact schemes without closed points.

And we conclude that $L_{X\setminus\{x_2,...,x_{n-1}\}} \cong \mathcal{O}(x_1)^{k_1}$. But then $L_{X\setminus\{x_2,...,x_{n-1}} \otimes \mathcal{O}(x_1)^{-k_1}$ is trivial on $X \setminus \{x_2, ..., x_{n-1}\}$ and we can inductively continue to deduce the claim. □

9. The first sheaf cohomology groups

Definition 9.1. (1) *Let G be a group. A G-torsor is a set P equipped with an action $G \times P \to P$ such that P is non-empty and this action is simply transitive, that is for each $p \in P$ we have that the map*

$$G \to P \qquad g \mapsto gp$$

is a bijection.

(2) *Given a space X and a sheaf of (not-necessarily abelian) groups $\mathcal{G}$ over X then another sheaf $\mathcal{P}$ (of sets) with an action*

$$\mathcal{G} \times \mathcal{P} \to \mathcal{P}$$

is a $\mathcal{G}$-torsor if all stalks of P are non-empty and for each $U \subseteq X$ and $p \in \mathcal{P}(U)$ the map

$$\mathcal{G}(U) \to \mathcal{P}(U) \qquad g \mapsto gp$$

is an isomorphism.

There is a (hopefully) obvious notion of morphism of torsors.

Example 9.2. (1) A torsor should be thought of as a copy of G but without a choosen basepoint. For example one could argue that our universe is more like a $(\mathbb{R}^3, +)$-torsor than $\mathbb{R}^3$ since there is no specificed basepoint.

(2) Let X be the Möbius strip and $\mathcal{G}$ be the constant sheaf $\mathbb{Z}/2$ and $\mathcal{P}$ be the sheaf of local orientations. Then $\mathcal{P}(X) = \emptyset$ but locally the action given by change of orientation is an isomorphism for every chosen orientation.

(3) Assume that we a vector bundle $\mathcal{V}$ on a scheme X of constant rank n. Consider the sheaf $\mathcal{P}$ of trivializations, i.e.

$$\mathcal{P}(U) = \{\varphi : \mathcal{O}_U^n \xrightarrow{\simeq} \mathcal{V}|_U \text{as } \mathcal{O}_U\text{-module sheaves}\} .$$

We have an action of the sheaf

$$\mathcal{G}(U) = \{\varphi : \mathcal{O}_U^n \xrightarrow{\simeq} \mathcal{O}_U^n \text{as } \mathcal{O}_U\text{-module sheaves}\}$$

of autormorphisms of $\mathcal{O}_U^n$ and an obvious action of $\mathcal{G}$ on $\mathcal{P}$ by composition. This makes the former a torsor over the latter.

(4) For every $\mathcal{G}$ there is the torsor given by $\mathcal{G}$ itself and the action given by left-multiplication. This is a torsor (the stalks are non-empty because they have a unit). We will refer to this as the trivial torsor.

(5) In general a torsor is isomorphic to the trivial torsor precisely of there is an element in $p \in \mathcal{P}(X)$ since then the map

$$\mathcal{G} \to \mathcal{P}$$

defined by

$$\mathcal{G}(U) \to \mathcal{P}(U) \qquad g \mapsto gp$$

is an isomorphism of torsors.

We claim that every map of $\mathcal{G}$-torsors is in fact an isomorphism. To see this we note that this is obviously true if we talk about classical (non-sheafify torsors), since maps are induced by (right) multiplication with elements of G. Then the statement about maps between sheaves can be checked on stalks which are classical torsors.
In fact, we see that a sheaf $\mathcal{P}$ with $\mathcal{G}$-action is a torsor precisely if all the stalks $\mathcal{P}_x$ are $\mathcal{G}_x$-torsors.

DEFINITION 9.3. *Given a space X and $\mathcal{G}$ a sheaf of groups, then*

$$H^1(X;\mathcal{G}) := \{\textit{isomorphism classes of } \mathcal{G}\textit{-torsors on } X\} \ .$$

is the first cohomology of X with coefficients in $\mathcal{G}$, which is in general just a set.

The set $H^1(X;\mathcal{G})$ comes with a canonical basepoint given by the trivial torsor. Actually, a priori it is not even clear that $H^1(X;\mathcal{G})$ is even a set.

LEMMA 9.4. *$H^1(X;\mathcal{G})$ is a set.*

PROOF. for every torsor $\mathcal{P}$ over X we find an open cover $U_i \subseteq X$ such that $\mathcal{P}(U_i) \neq \emptyset$. After choice of elements $p_i \in \mathcal{P}(U_i)$ we get trivializations $\mathcal{O}|_{U_i} \simeq \mathcal{G}|_{U_i}$. Then by descent for sheaves, Proposition 18.2, we see that the whole torsor is actually described by the isomorphisms $\varphi_{ij} : \mathcal{G}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ which are maps of torsors. These certainly form a set, so that for each fixed cover the set of all torsors that trivialize on that cover form a set and thus also the set of all torsors form a set. $\square$

REMARK 9.5. The proof also gives a different way of thinking about $H^1(X,\mathcal{G})$ which is closely related to Cech-cohomology (as will be discussed next term). Namely an element in $H^1(X,\mathcal{G})$ is describe by a pair consisting of a cover U_i of X and element $\varphi_{ij} \in \mathcal{G}(U_i \cap U_j)$ such that $\varphi_{ij} \cdot \varphi_{jk} = \varphi_{ik}$ in $\mathcal{G}(U_i \cap U_j \cap U_k)$. One can then identify when two such elements represent the same class in $H^1(X,\mathcal{G})$ by an explict description.

Assume that A is an abelian group. Then a left action of A on a set M can also be considered as a right action

$$m \cdot a := am$$

and we for two A-sets M and N we can form a new A-set

$$M \times_A N := (M \times N)/\simeq$$

with $(ma, n) \simeq (m, an)$. This is a A-set through left multiplication:

$$a \cdot [(m,n)] = (am, n) \ .$$

If M and N are A-torsors, then so is $M \times_A N$.
If $\mathcal{A}$ is a sheaf of abelian groups on X and $\mathcal{P}$ and $\mathcal{Q}$ are $\mathcal{G}$ torsors, then we define similarly

$$\mathcal{P} \times_{\mathcal{A}} \mathcal{Q}$$

by the sheafification of the presheaf

$$(\mathcal{P} \times_{\mathcal{A}} \mathcal{Q})(U) = \mathcal{P}(U) \times_{\mathcal{A}(U)} \mathcal{Q}(U) \ .$$

This is again a torsor.

LEMMA 9.6. *For $\mathcal{G}$ abelian the operation $(\mathcal{P},\mathcal{Q}) \mapsto \mathcal{P} \times_{\mathcal{A}} \mathcal{Q}$ endows $H^1(X,\mathcal{A})$ with the structure of an abelian group.*

PROOF. The fact that it is associated, unital and commutative are straighforward (checking the respective properties for $\mathcal{P} \times_{\mathcal{A}} \mathcal{Q}$. For the existence of inverses one simply takes a torsor $\mathcal{P}$ and equips it with a new $\mathcal{A}$-action obtained by action with the inverse map $- : \mathcal{A} \to \mathcal{A}$. Then one checks that this is indeed an inverse. Details omitted. $\square$

PROPOSITION 9.7. *Let $X = \operatorname{Spec}(R)$ be an affine scheme and $\mathcal{M}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules on M. Then*

$$H^1(X, \mathcal{M}) = 0 \ .$$

In order to prove this result we shall prove a more general result:

PROPOSITION 9.8. *Given a scheme X and a quasi-coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules, then there is a bijection:*

$$\begin{aligned} H^1(X, \mathcal{M}) &= \frac{\{\text{extensions of } \mathcal{O}_X\text{-modules } 0 \to \mathcal{M} \to \mathcal{M}' \to \mathcal{O}_X \to 0\}}{\text{isomorphism}} \\ &= \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \ . \end{aligned}$$

Here two extensions $0 \to \mathcal{M} \to \mathcal{M}' \to \mathcal{O}_X \to 0$ and $0 \to \mathcal{M} \to \mathcal{M}'' \to \mathcal{O}_X \to 0$ are isomorphic if there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{M}' \to \mathcal{M}''$ making the diagram

$$\begin{tikzcd} & & \mathcal{M}' \arrow[dd, "\simeq"] \arrow[dr] & & \\ 0 \arrow[r] & \mathcal{M} \arrow[ur] \arrow[dr] & & \mathcal{O}_X \arrow[r] & 0 \\ & & \mathcal{M}'' \arrow[ur] & & \end{tikzcd}$$

commutative.

REMARK 9.9. Note that every morphism $\mathcal{M}' \to \mathcal{M}''$ of extensions (as in the definition of isomorphismsm above) is automatically an isomorphism by the 5-lemma. Also note that all extensions $\mathcal{M}'$ are automatically quasi-coherent, since locally they split (by Lemma 2.10) so that locally $\mathcal{M}' = \mathcal{M} \oplus \mathcal{O}_X$.

PROOF OF PROPOSITION 9.8. We have a map

$$\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \to H^1(X, \mathcal{M})$$

which sends an extension $\mathcal{M} \xrightarrow{i} \mathcal{M}' \xrightarrow{p} \mathcal{O}_X$ to the sheaf

$$\mathcal{P} : U \mapsto \{s \in \mathcal{M}'(U) \mid p(s) = 1\}$$

which has an $\mathcal{M}$-action defined by:

$$\mathcal{M}(U) \times \mathcal{P}(U) \to \mathcal{P}(U) \qquad (m, s) \mapsto i(m) + s \ .$$

We claim that this is a torsor. To see this we note that locally around each point such splits exist (by Lemma 2.10), thus the stalks are non-empty. Moreover if we have given a split $s \in \mathcal{M}'(U)$ then this induces an isomorphism of the sequence to

$$\mathcal{M}|_U \to \mathcal{M}|_U \oplus \mathcal{O}_U \to \mathcal{O}_U$$

and we see that any further split is induced map a map $\mathcal{O}_U \to \mathcal{M}|_U$, i.e. a section of $\mathcal{M}|_U$. Therefore this is a torsor.

It is also clear that for isomorphic extensions the induced torsors are isomorphic so that we indeed get a well-defined map

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \to H^1(X, \mathcal{M})$$

which we will show to be a bijection.
Assume that we have two extensions $\mathcal{M} \to \mathcal{M}' \to \mathcal{O}_X$ and $\mathcal{M} \to \mathcal{M}'' \to \mathcal{O}_X$. Then we claim that morphisms (of extensions) $\mathcal{M}' \to \mathcal{M}''$ are 1-1 to morphisms of the associated torsors $\mathcal{P} \to \mathcal{P}'$. To see this we note that morpisms of extensions as well as morphisms of torsors are global sections of sheaves and our assignment defines a morphism of these sheaves. Thus to check that it is an isomorphisms we can check this locally, i.e. we can assume that our extensions are split, i.e. $\mathcal{M}' = \mathcal{M} \oplus \mathcal{O}_X$ and $\mathcal{M}'' = \mathcal{M} \oplus \mathcal{O}_X$. In this situation morphisms of extensions are given by morphisms

$$\mathcal{O}_X \to \mathcal{M}$$

i.e. global sections of $\mathcal{M}$. The same is true for morphisms of torsors and the assignment is also compatible
Thus we see that our map is injective. For subjectivity let $\mathcal{P}$ be a torsor over X. Choose a cover U_i on which it trivializes. Then we can find extensions which are preimages (namely the trivial ones). Moreover on double intersections we pick corresponding isomorphisms. Then these local choices glue to global ones using descent for sheaves as in Proposition 18.2. □

Remark 9.10. We have in fact shown more than $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \cong H^1(X, \mathcal{M})$ in the last proof, we have shown an equivalence of categories.

Proof of Proposition 9.7. For $X = \mathrm{Spec}(R)$ and $\mathcal{M} = \tilde{M}$ we have that extensions as in Proposition 9.8 are simply given by R-module extensions

$$0 \to M \to M' \to R \to 0$$

But since R is project all of those split so that $M' = M \oplus R$ is trivial. □

Let X be a topological space and $0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to 0$ an exact sequence of sheaves of abelian groups. Recall that this does not mean that

$$0 \to \mathcal{A}_1(X) \xrightarrow{i} \mathcal{A}_2(X) \xrightarrow{p} \mathcal{A}_3(X) \to 0$$

is exact, since the right hand map $\mathcal{A}_2(X) \to \mathcal{A}_3(X)$ is not necessarily an epimorphism (it is only so on stalks), so that we only have an exact sequence

$$0 \to \mathcal{A}_1(X) \xrightarrow{i} \mathcal{A}_2(X) \xrightarrow{p} \mathcal{A}_3(X) \ .$$

We claim that the cohomology H^1 can be seen as a way of measuring the deviation of this map to be surjective. Lets make this precise:

Proposition 9.11. *Let X be a topological space and $0 \to \mathcal{A}_1 \xrightarrow{i} \mathcal{A}_2 \xrightarrow{p} \mathcal{A}_3 \to 0$ an exact sequence of sheaves of abelian groups. Then the sequence above continues to an exact sequence*

$$0 \to \mathcal{A}_1(X) \xrightarrow{i} \mathcal{A}_2(X) \xrightarrow{p} \mathcal{A}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{A}_1) \xrightarrow{i} H^1(X, \mathcal{A}_2) \xrightarrow{p} H^1(X, \mathcal{A}_3)$$

of abelian groups.

We note that a version of this result is also true for sheaves of non-abelian groups where exactness is then interpreted in terms of pointed sets.

PROOF. We already have the first part. The maps $H^1(X, \mathcal{A}_1) \xrightarrow{i} H^1(X, \mathcal{A}_2) \xrightarrow{p} H^1(X, \mathcal{A}_3)$ are instances of the following construction: for a given morphism $f : \mathcal{G} \to \mathcal{G}'$ of sheaves of groups we get an induced morphism

$$H^1(X, \mathcal{G}) \to H^1(X, \mathcal{G}') \qquad \mathcal{P} \mapsto \mathcal{G}' \times_{\mathcal{G}} \mathcal{P}$$

where the latter is the sheafification of the sheaf with

$$(\mathcal{G}' \times_{\mathcal{G}} \mathcal{P})(U) = \mathcal{G}'(U) \times_{\mathcal{G}(U)} \mathcal{P}(U) \ .$$

It is not hard to see that this is indeed a torsor, defines a well-defined map as claimed. Moreover if both G and G' are abelian then this even a map of abelian groups. We also need to construct the map $\delta : \mathcal{A}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{A}_1)$. So given an element $s \in \mathcal{A}_3(X)$ we define a $\mathcal{A}_1$-torsor as follows:

$$U \mapsto \{s' \in \mathcal{A}_2(U) \mid p(s') = s|_U\}$$

with the action of $\mathcal{A}_1$-induced by the action of $\mathcal{A}_1$ on $\mathcal{A}_2$. It is not hard to check that this is again a torsor (using that the map $\mathcal{A}_2 \to \mathcal{A}_3$ is surjective on stalks). We leave it to the reader to check that δ is indeed a group homomorphism.
Now we want to check exactness at various spots. At the first two spots this is clear.
<u>Exactness of $H^1(X, \mathcal{A}_1) \xrightarrow{i} H^1(X, \mathcal{A}_2) \xrightarrow{p} H^1(X, \mathcal{A}_3)$</u>: Clearly the composition is zero, thus assume we have a torsor $\mathcal{P}$ for $\mathcal{A}_2$ and an isomophism $\mathcal{A}_3 \times_{\mathcal{A}_2} \mathcal{P} \cong \mathcal{A}_3$ of torsors. This gives a map of sheaves

$$f : \mathcal{P} \to \mathcal{A}_3$$

which is $\mathcal{A}_2$-equivariant (and the target carries the restricted action). Now we consider the sheaf

$$\ker(f) = \mathcal{P}' \subseteq \mathcal{P}$$

of all those elements in $\mathcal{P}$ which are mapped to the unit in $\mathcal{A}_3$, i.e.

$$\mathcal{P}'(U) = \{p \in \mathcal{P}(U) \mid f(p) = 0\} \ .$$

We claim that the $\mathcal{A}_2$-action on $\mathcal{P}$ restricts to a $\mathcal{A}_1$-action on $\mathcal{P}'$. To see this we have to note that for $a \in \mathcal{A}_1(U)$ and $p \in \mathcal{P}'(U)$ the element $f(ap) = [a]f(p) = f(p) = 0$. We get an induced map

$$\mathcal{P}' \times_{\mathcal{A}_1} \mathcal{A}_2 \to \mathcal{P}$$

induced by the inclusion. This map is an isomorphism, which can be checked stalkwise, where it is trivial. Thus the sheaf $\mathcal{P}'$ witnesses that our element indeed was in the image and we have established exactness.
<u>Exactness of $\mathcal{A}_2(X) \xrightarrow{p} \mathcal{A}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{A}_1)$</u>: We claim that the composition is trivial. To see this we consider $s \in \mathcal{A}_2(X)$ and then apply

$$\delta(ps)(U) = \{s' \in \mathcal{A}_2(U) \mid p(s') = (ps)|_U\}$$

This torsor has an obvious global section, namely s and thus is trivial. Now assume conversely that for a given $s \in \mathcal{A}_3(X)$ the torsor $\delta(s)$ has a global section $s' \in \mathcal{A}_s(X)$. Then this is by definition a lift of s so that indeed s is in the image. This shows exactness.
<u>Exactness of $\mathcal{A}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{A}_1) \xrightarrow{i} H^1(X, \mathcal{A}_2)$:</u> Let $s \in \mathcal{A}_3(s)$ and consider the torsor

$$\mathcal{A}_2 \times_{\mathcal{A}_1} \delta(s)$$

A section of this torsor is the same as a map $\delta(s) \to \mathcal{A}_2$ which is $\mathcal{A}_1$-equivariant. Such a map is provided by the obvious inlcusion

$$\delta(s)(U) = \{s' \in \mathcal{A}_2(U) \mid p(s') = (s)|_U\} \to \mathcal{A}_2(U) \ .$$

Conversely assume that we are given a $\mathcal{A}_1$-torsor $\mathcal{P}$ with a $\mathcal{A}_1$-equivariant map

$$f : \mathcal{P} \to \mathcal{A}_2$$

then we want to define an element $s \in \mathcal{A}_3(X)$. We will do define this locally: pick U small enough so that there is a section $s' \in \mathcal{P}(U)$. Then define

$$s|_U := pf(s') \in \mathcal{A}_3(X)$$

This element $s|_U$ does not depend on the choice of s' since another section differs by an element in $\mathcal{A}_1(U)$ and thus so do the images in $\mathcal{A}_2$ (since f is equivariant) and we conclude that they go to the same element under p. Once we know that $s|_U$ is well-defined we also see we get a global element s and we see that the morphism $f : \mathcal{P} \to \mathcal{A}_2$ induces a morphism

$$\mathcal{P}(U) \to \delta(s)(U) = \{s' \in \mathcal{A}_2(U) \mid p(s') = s\} \subseteq \mathcal{A}_2(U) \ .$$

This morphism is equivariant and thus an isosmorphism of torsors and therefore an isomorphism. This finishes the proof. $\square$

This results makes rigorous the way in which H^1 measures the deviation of global sections being exact. We will in fact see soon, that H^1 is also the universal such functor.

Example 9.12. Note that for an epimorphism $p : \mathcal{M}_2 \to \mathcal{M}_3$ of quasi-coherent sheaves on an *affine* scheme $X = \mathrm{Spec}(R)$ we know that the induced map $\mathcal{M}_2(X) \to \mathcal{M}_3(X)$ is indeed surjective. This is reflected by the fact that in this case the kernel $\mathcal{M}_1 = \ker(p)$ is quasi-coherent and $H^1(X, \mathcal{M}_1) = 0$ and thus the sequence

$$\mathcal{M}_2(X) \xrightarrow{p} \mathcal{M}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{M}_1) = 0$$

also shows the surjectivity. This surjectivity has been one of the very important facts which droves a lot of proves and statements in this course so far!

10. Higher sheaf cohomology groups

For an exact sequence $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ the map $H^1(X, \mathcal{A}_2) \xrightarrow{p} H^1(X, \mathcal{A}_3)$ is not surjective in general! The goal of sheaf cohomology is to further extend this construction to the right.

We now set $H^0(X, \mathcal{A}) = \mathcal{A}(X)$. Lets summarize our findings so far:

(1) Any morphism of sheaves $f : \mathcal{A} \to \mathcal{B}$ induces morphisms f_*:

$$H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \qquad \text{and} \qquad H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B})$$

This is a functor, i.e. $(fg)_* = f_* g_*$ and $\mathrm{id}_* = \mathrm{id}$.

(2) for any short exact sequence $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ we get a natural map

$$\delta : H^0(X, \mathcal{A}_3) \to H^1(X, \mathcal{A}_1) \ .$$

Naturality hear means natural in short exact sequences, where morphisms of such are 'ladder' diagrams.

(3) The associated sequence

$$0 \to \mathcal{A}_1(X) \xrightarrow{i} \mathcal{A}_2(X) \xrightarrow{p} \mathcal{A}_3(X) \xrightarrow{\delta} H^1(X, \mathcal{A}_1) \xrightarrow{i} H^1(X, \mathcal{A}_2) \xrightarrow{p} H^1(X, \mathcal{A}_3)$$

is exact.

(4) If X is an affine scheme and $\mathcal{M}$ quasi-coherent then $H^1(X, \mathcal{M}) = 0$.

The goal of sheaf cohomology is to further extend this sequence, that is to construct abelian groups

$$H^i(X, \mathcal{A}) \qquad \text{for } i \in \mathbb{N}$$

such that analogous of these properties hold and such that it is universal with respect to those.

Definition 10.1. *A δ-functor (from sheaves on X to abelian groups) is given by a sequence of product preserving functors*

$$F^i : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab} \qquad i \in \mathrm{N}$$

together with a natural maps

$$\delta^i : F^i(\mathcal{A}_3) \to F^{i+1}(\mathcal{A}_1)$$

for every short exact sequence $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ such that the induced sequence

$$0 \to F^0(\mathcal{A}_1) \to F^0(\mathcal{A}_2) \to F^0(\mathcal{A}_3) \to F^1(\mathcal{A}_1) \to ..$$

is long exact. A morphism of delta functors $(F^i, \delta^i) \to (G^i, \delta^i)$ is given by a sequence of natural transformations $F^i \to G^i$ such that the diagram

$$\begin{tikzcd} F^i \arrow[r] \arrow[d, "\delta"] & G^i \arrow[d, "\delta"] \\ F^{i+1} \arrow[r] & G^{i+1} \end{tikzcd}$$

commutes.

The main result about sheaf cohomology is the following:

Theorem 10.2 (Grothendieck). *For every space X there exists an initial δ-functor (H^i, δ^i) with $H^0(\mathcal{A}) \cong \mathcal{A}(X)$. We write*

$$H^i(X, \mathcal{A})$$

and call it the i-th sheaf cohomology of X with value in $\mathcal{A}$.
Moreover in degree 1 it agrees with our H^1 and δ^0 agrees with our δ. For affine schemes X and quasi-coherent sheaves $\mathcal{A}$ we have that $H^i(X, \mathcal{A}) = 0$ for $i > 0$.

Note that being initial of course uniquely determines this functor. We will not fully prove this result, but give an idea how it works. For this the notion of an injective object is central:

Definition 10.3. *An object $A \in \mathcal{C}$ in a category is called* injective *if every monomorphism $i : A \to B$ has a retract, i.e. a map $r : B \to A$ sich that $ri = \mathrm{id}_A$.*

Example 10.4. (1) every set is injective in the category of sets

(2) Every vector space over a field k is injective. To see this assume hat $V \to V'$ is an injection. Then we choose a complement to V in V' and take the retraction.

(3) The abelian group $\mathbb{Z}$ is not injective, since $\mathbb{Z} \to \mathbb{Q}$ does not admit a section. The groups $\mathbb{Z}/n$ is also not injective since the inclusion $\mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ sending 1 to $1/n$ does not admit a section.

(4) The abelian group $\mathbb{Q}$ is injective (without proof). More generally an abelian group A is injective, precisely if it is divisible, that is if for every $a \in A$ and $n \in \mathbb{Z}$ there exists an object b with $nb = a$ (also without proof).

LEMMA 10.5. *Let $\mathcal{F}$ be an injective sheaf of abelian groups on a space X. Then $H^1(X, \mathcal{F}) = 0$.*

PROOF. We first claim that for an arbitrary sheaf of abelian groups $\mathcal{F}$ there is a 1-1 correspondence between elements in $H^1(X, \mathcal{F})$ and isomorphisms classes of extensions

$$0 \to \mathcal{F} \to \mathcal{F}' \to \underline{\mathbb{Z}} \to 0$$

where $\underline{\mathbb{Z}}$ is the sheaf of locally constant $\mathbb{Z}$-valued functions. This 1-1 correspondence follows exactly as the one in Proposition 9.8. Moreover under this correspondence the trivial $\mathcal{F}$-torsor corresponds to the split extension. Now if $\mathcal{F}$ is injective then such an extension splits automatically since the map $\mathcal{F} \to \mathcal{F}'$ admits a retract. $\square$

THEOREM 10.6 (Enough injectives). *Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. Then there exists a monomorphism $\mathcal{F} \to \mathcal{F}'$ where $\mathcal{F}'$ is injective.*

The last two statements together already detemine $H^1(X, \mathcal{A})$ for a sheaf of abelian groups $\mathcal{A}$: we embedd $\mathcal{A}$ into an injective sheaf $\mathcal{I}$ and form the short exact sequence

$$\mathcal{A} \to \mathcal{I} \to \mathcal{I}/\mathcal{A}$$

which induces a long exact sequence:

$$\mathcal{I}(X) \to (\mathcal{I}/\mathcal{A})(X) \to H^1(X, \mathcal{A}) \to H^0(X, \mathcal{I}) = 0$$

so that we see that $H^1(X, \mathcal{A})$ is the cokernel of $\mathcal{I}(X) \to (\mathcal{I}/\mathcal{A})(X)$ for any monomorphism $\mathcal{A} \to \mathcal{I}$ with $\mathcal{I}$ injective.
Now the main idea to prove 9.8 and to compute the sheaf cohomology is to prove two things:

(1) Construct a δ-functor H^i with $H^0(\mathcal{A}) = \mathcal{A}(X)$ and $H^i(\mathcal{I}) = 0$ for $\mathcal{I}$ injective and $i > 0$.

(2) Prove that any δ-functor which vanishes on injectives in positive degrees is already initial.

As a result we see that $H^i(X, \mathcal{I}) = 0$ and we can apply a trick similar to above to use injective resolutions to compute the sheaf cohomology.

11. Outlook: the Riemann–Roch theorem

DEFINITION 11.1. *If A is an algebra over any base ring k and M is an A-module then a k-linear map $d : A \to M$ is called a k-linear derivation if $d(ab) = a \cdot d(b) + d(a) \cdot b$.*

Of course in this case we have $d(1) = 0$ and thus $d(\lambda) = 0$ for all $\lambda \in k$. As an example let $A = k[t]$ and $M = A$. Then the usual derivative

$$k[t] \to k[t] \qquad p \mapsto p'$$

is a derivation.

Proposition 11.2. *For any k-algebra A there is an initial k-linear derivation*

$$d : A \to \Omega^1_{A/k}$$

This is for any other derivation $d' : A \to M$ there is a unique k-linear morphism $f : \Omega^1_{A/k} \to M$ such that $fd = d'$.

The module $\Omega^1_{A/k}$ is called the module of Kähler differentials.

Proof. We define $\Omega^1_{A/k}$ as the A-module freely generated by symbols da for $a \in A$ subject to the relations

$$d(ab) = ad(b) + bd(a) \qquad d(k) = 0 \qquad d(a+b) = da + db \ .$$

Then the ma[

$$A \to \Omega^1_{A/k}$$

is a k-linear derivation and it is clear that $\Omega^1_{A/k}$ is initial. □

Example 11.3. For $A = k[t]$ we have that

$$\Omega^1_{k[t]/k} = k[t] \cdot dt$$

is a free module on a single generator (called dt).

Proposition 11.4. *The construction $A \mapsto \Omega^1_{A/k}$ commutes with localizations, that is*

$$\Omega^1_{A[b^{-1}]/k} \cong \Omega^1_{A/k} \otimes_A A[b^{-1}] \ .$$

In particular we obtain for every scheme X a quasi-coherent sheaf $\Omega^1_{X/k}$ on X such that the value on affine opens $\mathrm{Spec}(A) \subseteq X$ *is given by the Kähler-differentials of A.*

Let k be an algebraically closed field, and C a projective and smooth curve over k (the latter notion has not been defined yet). Then one can show that $\Omega^1_{C/k}$ is a line bundle.

Definition 11.5. *Let C be projective and smooth curve over k, then the genus g is defined as*

$$g := \dim_k(\Gamma(C, \Omega^1_{C/k}))$$

Theorem 11.6 (Riemann–Roch Theorem). *For all line bundles $\mathcal{L}$ on C we have*

$$\dim_k \Gamma(C, \mathcal{L}) - \dim_k \Gamma(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g$$

The proof strategy will rely crucially on sheaf cohomology. In fact we will show two things:

(1) For any line bundle $\mathcal{L}$ on C we have

$$\dim_k H^0(C, \mathcal{L}) - \dim_k H^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

and the left hand terms are finite (which is in some sense the hardest part).

(2) For a proper, smooth scheme X over a field k of dimension d, and a vector bundle $\mathcal{E}$ over X, then we have an isomorphism

$$H^i(X, \mathcal{E}) \cong H^{d-i}(X, \Omega^d_{X/k} \otimes \mathcal{E}^\vee)^\vee$$

This is Serre duality. In particular in our situation for $X = C$ we have $d = 1$ so that $H^1(C, \mathcal{L})$ is dual over k to $H^0(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee)$ and thus has the same dimension.

CHAPTER 3

Algebraic Geometry III, Wintersemester 2022/23

1. Overview: Smoothness and Riemann–Roch

In the last terms we have talked about schemes and varieties. Let us review some basic notation first.

Recall that for a scheme X we have the structure sheaf $\mathcal{O}_X$ and the local rings $\mathcal{O}_{X,x}$ for each point $x \in X$. Then the residue field at x is given by $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Locally every scheme by definition looks like $X = \mathrm{Spec}(R)$ in which case we have for a point $x \in Spec(R)$ corresponding to a prime ideal that

$$\mathcal{O}_{\mathrm{Spec}(R),x} = R_x$$

is the localization at x, i.e. inverting all elements in the complement of x. The maximal ideal is given by $\mathfrak{m}_x = xR_x \subseteq R_x$ and the residue field by

$$\kappa(x) = R_x/xR_x = \mathrm{Frac}(R/x) \ .$$

We have then discussed the notion of quasi-coherent sheaves on schemes X. If $X = \mathrm{Spec}(R)$ then quasi-coherent sheaves on X are given by $\mathcal{M} = \tilde{M}$ for R-modules M, in fact there is an equivalence of categories $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R$. For every quasi-coherent sheaf $\mathcal{M}$ we have the stalk $\mathcal{M}_x$ which is a module over $\mathcal{O}_{X,x}$ and then the fibre

$$\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \ .$$

This is also the same as the pullback of $\mathcal{M}$ to $\mathrm{Spec}(\kappa(x))$ along the canonical morphism $\mathrm{Spec}(\kappa(x)) \to X$ which factors as

$$\mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X \ .$$

For a given module M we have that

$$\mathcal{M}_x = M_x$$

and the fibre is

$$M_x/xM_x \cong (M/xM) \otimes_{R/x} \kappa(x) \ .$$

Example 1.1. Let $X = \mathrm{Spec}(\mathbb{Z})$. Then the points are given by primes p and the generic point 0 with residue fields $\kappa(p) = \mathbb{F}_p$ and $\kappa(0) = \mathbb{Q}$. A quasi-coherent sheaf is simply and abelian group M and then the fibres are given by the quotients M/p (considered as $\mathbb{F}_p$ vector space) and the rationalization $M \otimes \mathbb{Q}$.

Let k be algebraically closed (or maybe $k = \mathbb{C}$ for intuition). A variety is a scheme over k which is reduced, irreducible, separated and of finite type. In this case we have that all the residue fields $\kappa(x)$ are field extensions of K of finite transcendence degree (of course there are not finite field extensions). For closed points the residue fields are isomorphic to k itself. Recall that affine varieties are the same as irreducible affine algebraic sets, i.e. zeros of finitely many polynomials in n variables over k which are additionally irreducible. Examples of (affine varieties) include curves like

(1) $V(y + 3x - 4)$ (a straight line)
(2) The elliptic curve $y^2 = x^3 - x + 1$ which is smooth (at least over $\mathbb{R}$ as a smooth manifold)
(3) Another elliptic curve is $y^2 = x^3 - x$ which is also smooth but disconnected over $\mathbb{R}$.
(4) The elliptic curve $y^2 = x^3 - x^2$ has a "nodal singularity" at the origin.
(5) The curve $y^2 = x^3$ has a "cuspidal singularity" at the origin.

One of the topics of this course is to make rigorous the notion of "smoothness" in the algebraic setting and define what a "singularity" is. For varieties smootheness is equivalent to regularity, which we will motivate now: the rough idea is to define a tangent space to schemes X for every point $x \in X$.

Definition 1.2. *This cotangent space T_x^*X is defined in terms of the local ring $\mathcal{O}_{X,x}$ as*

$$T_x^*X := \mathfrak{m}_x/\mathfrak{m}_x^2 \ .$$

This quotient is a quotient of $\mathcal{O}_{X,x}$-modules. But it will be a module over $\mathcal{O}_{X,x}/\mathfrak{m}_x$ since $\mathfrak{m}_x$ acts trivially. Thus the cotangent space at x is a $\kappa(x)$-vector space.
We will see that these vector spaces can be assembled together into a quasi-coherent sheaf $\Omega^1_{X/k}$ on X, i.e. we have for all closed points $x \in X$ that

$$T_x^*X = \Gamma(i_x^*(\Omega^1_{X/k})) = \Omega^1_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \ .$$

where $i_x : \mathrm{Spec}(\kappa(x)) \to X$ is the canonical map induced by the point x. This sheaf $\Omega^1_{X/k}$ will be refered to as the sheaf of Kähler differentials of X or the cotangent sheaf of X.

Definition 1.3. *The tangent space at x is simply defined as the dual space*

$$T_xX := (T_x^*X)^\vee = \mathrm{Hom}_{\kappa(x)}(T_x^*X, \kappa(x))$$

We will see that this tangent space in the above examples pretty much encapsulates the idea of a tangent space (and is closely related to the tangent space we know for smooth manifolds). In fact, it turns out that a tangent vector at a closed point x of a variety i.e. an element of the tangent space T_xX is essentially the same datum as a map

$$\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to X$$

over and under the map $i_x : \mathrm{Spec}(k) \to X$ given by the point $x \in X$. Note that the ring $k[\epsilon]/\epsilon^2$ is not reduced, so $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ is far from being a variety.

Definition 1.4. *We say that a variety X is smooth at $x \in X$ if we have that*

$$\dim \mathcal{O}_x = \dim_{\kappa(x)} T_xX \ .$$

Here the first means the Krull dimension.

In the examples above the Krull dimension of all the local rings is 1 (these are curves after all) but the tangent dimension can be higher, namely in the "singular cases". We will different characterisations of smootheness in the course. One of the consequences of smootheness is that the Kähler differentials $\Omega^1_{X/k}$ form a vector bundle whose rank is the Krull dimension of X. In particular for curves C the Kähler differentials for a line bundle.

1.1. Riemann–Roch.

DEFINITION 1.5. *Let C be projective and smooth curve over k, then the genus g is defined as*

$$g := \dim_k(\Gamma(C, \Omega^1_{C/k})) \ .$$

Note that it is a priori not even clear that this is a finite number (but it turns out to be as we will see). Informally this measures the number of 'holes' in the picture of curves over $\mathbb{C}$ as surfaces, i.e. elliptic curves have genus 1 and $\mathbb{P}^1$ has genus zero.

The main result that we will prove and one of the most important results in algebraic geometry is the following:

THEOREM 1.6 (Riemann–Roch Theorem). *For all line bundles $\mathcal{L}$ on C we have*

$$\dim_k \Gamma(C, \mathcal{L}) - \dim_k \Gamma(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g$$

The proof strategy will rely crucially on sheaf cohomology. Thus a considerable part of this course will be about sheaf cohomology. Sheaf cohomology takes the following form: for any topological space X and every sheaf $\mathcal{F}$ on X we define abelian groups

$$H^i(X, \mathcal{F}) \qquad i \in \mathrm{N}$$

called sheaf cohomology groups. If $\mathcal{F}$ is a sheaf of k-vector spaces, then the groups $H^i(X, \mathcal{F})$ are also k-vector spaces. The design criterion is that $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ and that for every short exact sequence of sheaves

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2$$

we get a long exact sequence

$$0 \to H^0(X, \mathcal{F}_0) \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \xrightarrow{\delta} H^1(X, \mathcal{F}_0) \to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2) \to H^2(X, \mathcal{F}_0)$$

So in some sense the groups $H^i(X, \mathcal{F})$ measure the deviation of exactness of global sections. Sheaf cohomology is very important way beyond the scope of this course!

After establishing the framework for sheaf cohomology we will prove the following facts, which imply Riemann-Roch:

(1) For any line bundle $\mathcal{L}$ on C we have

$$\dim_k H^0(C, \mathcal{L}) - \dim_k H^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

and the left hand terms are finite (which is in some sense the hardest part).

(2) For a projective, smooth scheme X over a field k of dimension d, and a vector bundle $\mathcal{E}$ over X, then we have an isomorphism

$$H^i(X, \mathcal{E}) \cong H^{d-i}(X, \Omega^d_{X/k} \otimes \mathcal{E}^\vee)^\vee$$

This is Serre duality. In particular in our situation for $X = C$ we have $d = 1$ so that $H^1(C, \mathcal{L})$ is dual over k to $H^0(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee)$ and thus has the same dimension.

Literature that I have used to prepare the course is:

(1) R. Hartshorne: Algebraic Geometry GTM 52. Springer.
(2) D. Mumford: The red book of varieties and schemes. Springer LN 1358.
(3) U. Goertz, T. Wedhorn: Algebraic Geometry I. Vieweg.
(4) A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique.

(5) P. Scholze: Algebraic Geometry I & II, lecture notes (typed by Jack Davies)
(6) The stacks project

We will assume knowledge about the basics of scheme theory: the notion of a scheme, morphisms of schemes, the notion of quasi-coherent sheaves, line bundles, vector bundles. The most important example for us is projective space $\mathbb{P}^n$ and we encourage everyone to understand this scheme from as many perspectives as possible. These preliminaries are all covered in the first part of the course.
Further notions that will sometimes play a role are

(1) Closed immersions (Theorem about affine situation)
(2) pushforward and pullback of quasi-coherent sheaves
(3) Noetherian schemes and coherent modules
(4) Dimension theory, Noether normalization
(5) finite type morphisms
(6) Varieties and Projective varieties

2. Flatness

Definition 2.1. *Let R be a ring. Then an R-module M is flat if the functor $-\otimes_R M$ is exact. Equivalently if for every mononorphsm of R-modules $N \to N'$ the induced map*

$$N \otimes_R M \to N' \otimes_R M$$

is also injective.
A map $R \to S$ of rings is called flat *if S is flat as an R-module. It is called* faithfully flat *if additionally the induced map* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *is surjective.*

Example 2.2. (1) Free modules are flat, since tensoring is just a direct sum.
(2) Projective modules are flat, since retracts of flat modules are clearly flat.
(3) The $\mathbb{Z}$-module $\mathbb{Z}/2$ is not flat, since the short exact sequence

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2$$

becomes after tensoring with $\mathbb{Z}/2$ the seuqence

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\mathrm{id}} \mathbb{Z}/2$$

which is not exact.
(4) The localization of a ring $R \to R[S^{-1}]$ at a set S is flat. This follows since tensoring with $R[S^{-1}]$ is the functor

$$N \mapsto N[S^{-1}]$$

which preserves monomorphisms since for a given monomorphism $i : N \to N'$ an element n/s in the kernel of $N[S^{-1}] \to N'[S^{-1}]$ has to have the property that $i(n)$ is anihilated by some s, i.e. $s'i(n) = 0$. But this implies that $i(sn) = 0$ and thus $sn = 0$.
(5) In general filtered colimits of flat modules are flat, since filtered colimits are exact. For example we can write $R[x^{-1}]$ also as the filtered colimit

$$R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to ....$$

(6) Assume that R is an $\mathbb{F}_p$-algebra. Then the Frobenius $R \to R$ is a ring map which induces the identity map $\operatorname{Spec}(R) \to \operatorname{Spec}(R)$, since the pullback of a prime ideal is the same prime ideal. It follows that the Frobenius is

faithfully flat if and only of it is flat. Such $\mathbb{F}_p$-algebras are closely related to regular rings by a theorem of Kunz.

LEMMA 2.3. *For an R-module M the following are equivalent:*

(1) *M is flat*
(2) *For sequences $N \to N' \to N''$ exact in the middle, the induced sequences after tensoring with M is exact in the middle.*
(3) *for every ideal $I \subseteq R$ the induced map*
$$M \otimes_R I \to M \otimes_R R = M$$
is injective. [1]
(4) *for every finitely generated deal $I \subseteq R$ the induced map*
$$M \otimes_R I \to M \otimes_R R = M$$
is injective

PROOF. Omitted since tedious (but elementary), see Lemma 00HD. □

COROLLARY 2.4. *A $\mathbb{Z}$-module is flat precisely if it is torsion free.*

PROOF. For the ideals $n\mathbb{Z} \subseteq \mathbb{Z}$ the induced map is given by $M \xrightarrow{n} M$ which is injective precisely if M is n-torsion free. □

THEOREM 2.5 (Lazard). *An R-module is flat if and only if it is a filtered colimit of finite free modules.*

PROOF. Let M be an R-module which is flat. We have to show that it is a filtered colimit of free ones. To this end we note that M is the colimit of all finite free modules mapping to M. In other words, it is the colimit of the diagram
$$\mathrm{Free}_{/M} \to \mathrm{Mod}_R$$
which sends $F \to M$ to F. Thus it suffices to show that $\mathrm{Free}_{/M}$ is indeed filtered. Clearly this slice has coproducts (given by the sum of free modules) this to show that it is filtered, it suffices to check that any pair of parallel morphisms.
$$f, g : F_1 \to F_2$$
in $\mathrm{Free}_{/M}$ can be equalized, i.e. there is a map $p : F_2 \to F_3$ such that $pf = pg$. Note that we cannot simply take the coequalizer, since this wouldn't necessarily by free. What we instead do is to consider the dual maps
$$(F_2)^\vee \xrightarrow{f^\vee, g^\vee} (F_1)^\vee$$
and find a free module $F_3 = \oplus_I R$ with a surjective map h to the equalizer of $f^\vee$ and $g^\vee$, i.e. the kernel of the difference, i.e. we find a sequence
$$\mathrm{colim}_{I' \subseteq I \text{finite}} R^{I'} \xrightarrow{h} F_2^\vee \xrightarrow{f^\vee - g^\vee} F_1^\vee$$
which is exact in the middle. Then the resulting sequence after tensoring with M will also be exact in the middle:
$$\mathrm{colim}_{I' \subseteq I \text{finite}} \mathrm{Hom}(R^{I'}, M) \xrightarrow{(h_{I'}^\vee)^*} \mathrm{Hom}(F_2, M) \xrightarrow{(f-g)^*} \mathrm{Hom}(F_1, M)$$
Thus applying this to the structure maps $F_2 \to M$ we find a map $R^{I'} \to M$ such that $(h_{I'}^\vee) : F_2 \to R^{I'}$ equalizes the maps. □

[1]One could equivalently phrase this as saying that $M \otimes_R I \to IM$ is an isomorphism.

PROPOSITION 2.6. *Let R be noetherian with an ideal $I \subseteq R$. Then the 'completion'*

$$R \to R_I^\wedge = \varprojlim R/I^n$$

is flat.

PROOF. Let $J \subseteq R$ be another ideal. We claim that $J \otimes_R R_I^\wedge \to R_I^\wedge$ is injective. This follows from the following two facts:

- $J \otimes_R R_I^\wedge = J_I^\wedge$
- For an injection $M \to N$ of finitely generated R-modules the induced map $M_I^\wedge \to N_I^\wedge$ is injective.

which we also omit (see stacks project Lemma 00MA) □

EXAMPLE 2.7. Assume that $f_1, .., f_n \in R$ are such that $1 \in (f_1, ..., f_n)$ or equivalently the $D(f_i)$ form a cover of $\mathrm{Spec}(R)$. Then the map

$$R \to \prod R[1/f_i]$$

is faithfully flat. Indeed, it is flat as a finite sum of flat modules and the covering property exactly implies surjectivtity.

LEMMA 2.8. *If $f : R \to S$ is faithfully flat and for an R-module M we have $M \otimes_R S = 0$ then $M = 0$.*

PROOF. Assume not and pick $x \in M$ non-trivial. Let

$$I = \{r \in R \mid rx = 0\} \subsetneq R$$

be the annihilitor ideal of x. Then the map

$$R/I \to M \qquad [r] \mapsto rx$$

is an injective map of R-modules, so that after tensoring with S the induced map

$$S/IS \to M \otimes_R S = 0$$

is also injective, so that $S/IS = 0$. On the other hand we have a pullack diagram

$$\begin{tikzcd} \mathrm{Spec}(S/IS) \arrow[r] \arrow[d] & \mathrm{Spec}(R/I) \arrow[d] \\ \mathrm{Spec}(S) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

of schemes. The lower map is surjective and the vertical maps are closed immersions. In this situation the top map must also be surjective (think about what prime ideals in R/I and S/IS are...). Concretely we see that prime ideal in R/I are simply prime ideals in R which contain I and simillarly form S/IS. The upper map being surjective is in contradiction to to $S/IS = 0$ (i.e. $\mathrm{Spec}(S/IS) = \emptyset$) but $R/I \neq 0$ (i.e. $\mathrm{Spec}(R/I) \neq \emptyset$). □

LEMMA 2.9. *Let $R \to S$ be a map of rings. Then $- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ sends flat modules to flat modules. If $R \to S$ is flat then restriction $\mathrm{Mod}_S \to \mathrm{Mod}_R$ also preserves flat modules.*

PROOF. Let M be a flat R-module and $N \to N'$ be a monomorphism of S-modules. Then $(M \otimes_R S) \otimes_S N = M \otimes_R N$ and similary for N'. Thus flatness follows from flatness of M.

For the second part assume that M is a flat S-module and $R \to S$ is flat. Then for every injection

$$N \to N'$$

R-modules. We have

$$N \otimes_R M = (N \otimes_R S) \otimes_S M$$

and similar for N'. Thus flatness if S implies that $N \otimes_R S \to N' \otimes_R S$ is injetive and then flatness of M implies that $N \otimes_R M \to N' \otimes_R M$ is injective. $\square$

Proposition 2.10. *(Flatness is flat-local) Assume that $R \to S$ is faithfully flat and M is an R-module. Then M is flat iff $M \otimes_R S$ is flat as an S-module.*

Proof. By the previous Lemma we we only need to show the 'if' direction. To this end assume that $M \otimes_R S$ is flat and we are given an injection $N \to N'$ of R-modules. Let K be the kernel of $N \otimes_R M \to N' \otimes_R M$ and consider the exact sequence

$$0 \to K \to N \otimes_R M \to N' \otimes_R M$$

of R-modules. After basechanging to S we get by flateness an exact sequnence

$$0 \to K \otimes_R S \to N \otimes_R M \otimes_R S \to N' \otimes_R M \otimes_R S$$

By flatness of $M \otimes_R S$ and the previous Lemma the right hand map is injective. This $K \times_R S = 0$ which by Lemma 2.8 implies that $K = 0$. $\square$

Corollary 2.11 (Flatness is a Zariski-local property). *Let M be an R-module and $R_i = R[f_i^{-1}]$ be a Zariski cover of R. Then M is flat iff and only if $M \otimes_R R_i$ is flat for all i.*

Corollary 2.12. *Let $\mathcal{M}$ be a quasi-coherent sheaf on a scheme X. Then TFAE;*

- *For all open affines $U = \mathrm{Spec}(R) \subseteq X$ the R-module $\mathcal{M}(U)$ is flat.*
- *There exists an affine cover $U_i = \mathrm{Spec}(R_i)$ such that $\mathcal{M}(U_i)$ are flat R_i-modules.*

Proof. We only need to show that the second condition implies the first. Thus assume that $U_i = \mathrm{Spec}(R_i)$ is a cover as in the second and $U = \mathrm{Spec}(R)$ is affine. Then we can form the intersections

$$U \cap U_i$$

These need not be affine, but we can find further refinements $V_{ij} \subseteq U \cap U_i$ which are affine and which are principal open in U. We first observe that $\mathcal{M}(V_{ij})$ is flat (by Lemma 2.9 since it is a basechange of $\mathcal{M}(U_i)$ as $\mathcal{M}$ is quasi-coherent). Therefore we have found a cover of U by principal opens on which $\mathcal{M}$ is flat, thus by Corollary 2.11 we get that $\mathcal{M}(U)$ is flat. $\square$

Definition 2.13. *If one of these conditions is satisfied then we call $\mathcal{M}$ flat. A morphism $Y \to X$ of schemes is called flat if for each affine open $\mathrm{Spec}(S) = U \subseteq Y$ mapping to an affine open $\mathrm{Spec}(R) = V \subseteq X$ the induced map $R \to S$ is flat.*

Proposition 2.14. *A morphism $Y \to X$ is flat iff there exists an affine open cover $U_i = \mathrm{Spec}(R_i) \subseteq Y$ mapping to open affines $V_i = \mathrm{Spec}(S_i) \subseteq X$ such that R_i is flat over S_i.*

PROOF. One direction is obvious. For the other one we assume that the conditions of the proposition are satisfied and want to show that $Y \to X$ is flat. We first claim that if $\mathrm{Spec}(R_i') = U_i' \subseteq U_i$ and $\mathrm{Spec}(S_i) = V_i' \subseteq V_i$ with $f(U_i') \subseteq V_i'$ then R_i' is flat over S_i'. This indeed follows since S_i' is the basechange of S_i along $R_i \to R_i'$. Therefore we can refine the cover for which the condition of the proposition holds. $\square$

Now we want to understand faithfully flat descent. Recall the descent statement that we proved for modules (in the first part of the course):

PROPOSITION (Descent for modules). *Let R be a ring with a principal open cover $(R_i = R[f_i^{-1}])_{i\in I}$ (i.e. $\mathrm{Spec}(R_i)$ an open cover of $\mathrm{Spec}(R)$). Then the functor*

$$\begin{aligned}\mathrm{Mod}_R &\to \mathrm{Desc}_{(R_i)}(\mathrm{Mod}) \\ &= \{M_i \in \mathrm{Mod}_{R_i}, \quad \varphi_{ij} : M_i[f_j^{-1}] \xrightarrow{\simeq} M_j[f_i^{-1}] \mid \textit{cocycle condition}\}\end{aligned}$$

is an equivalence of categories.

Now we want to formulate a more general version of this result for arbitrary faithfully flat maps $f : R \to S$ replacing the open cover

$$f : R \to \prod R_i$$

The result is the following:

THEOREM 2.15 (Faithfully flat descent). *Let $f : R \to S$ be a faithfully flat map. Then the functor*

$$\begin{aligned}&\mathrm{Mod}_R \to \mathrm{Desc}_f(\mathrm{Mod}) \\ &\quad = \{N \in \mathrm{Mod}_S, \quad \varphi : N \otimes_R S \xrightarrow{\simeq} S \otimes_R N \textit{ as } S \otimes_R S\textit{-modules} \mid \textit{cocycle condition}\}\end{aligned}$$

sending M to $S \otimes_R M$ with the 'canonical' choice of φ is an equivalence of categories.

Here the cocycle condition is the condition that we have a commutative diagram

$$\begin{tikzcd} N \otimes_R S \otimes_R S \arrow[r, "\varphi\otimes\mathrm{id}_S"] \arrow[dr, "\varphi_{(1)}\otimes\mathrm{id}_S\otimes\varphi_{(2)}"'] & S \otimes_R N \otimes_R S \arrow[d, "\mathrm{id}_S\otimes\varphi"] \\ & S \otimes_R S \otimes_R N \end{tikzcd}$$

Here the diagonal map can also be seen as flipping the first two factors, applying $\mathrm{id} \otimes \varphi$ and switching the first two factors again. The descent category $\mathrm{Desc}_f(\mathrm{Mod})$ becomes a category in the (hopefully) obvious way allowing morphisms in Mod_S which preseve the descent datum.

The functor $\mathrm{Mod}_R \to \mathrm{Desc}_f(\mathrm{Mod})$ sends M to the S-module $N = S \otimes_R M$ with the isomorphism

$$\varphi : (S \otimes_R M) \otimes_R S \cong S \otimes_R (S \otimes_R M)$$

given by flipping the second and third factor. This isomorphism satisfies the cocycle condition because the diagram takes the form

$$\begin{tikzcd} S \otimes_R M \otimes_R S \otimes_R S \arrow[r] \arrow[dr] & S \otimes_R S \otimes_R M \otimes_R S \arrow[d] \\ & S \otimes_R S \otimes_R S \otimes_R M \end{tikzcd}$$

in which the upper horizontal map switches the middle factors, the right vertical map switches the right most factors and the diagonal map switches the M with the last factor.
We encourage the reader to work out how Zariski descent (as stated above) is a special case. Now we will provide some results that will eventually lead to a proof of Theorem 2.15.

LEMMA 2.16. *A morphism $f : (N, \varphi) \to (N', \varphi')$ in $\mathrm{Desc}_f(\mathrm{Mod})$ is an isomorphism iff the underlying morphism $N \to N'$ of S-modules is an isomorphism.*

PROOF. By definition the morphism $f : N \to N'$ makes the diagram

$$\begin{tikzcd} N \otimes_R S \arrow[r, "\varphi"] \arrow[d, "f\otimes \mathrm{id}"] & S \otimes_R N' \arrow[d, "\mathrm{id}\otimes f"] \\ N' \otimes_R S \arrow[r, "\varphi'"] & S \otimes_R N' \end{tikzcd}$$

commutative. Thus its inverse f^{-1} makes the diagram

$$\begin{tikzcd} N \otimes_S R \arrow[r, "\varphi"] & R \otimes_S N \\ N' \otimes_S R \arrow[r, "\varphi'"] \arrow[u, "f^{-1}\otimes \mathrm{id}"] & R \otimes_S N \arrow[u, "\mathrm{id}\otimes f^{-1}"'] \end{tikzcd}$$

commutative. This shows the claim. $\square$

LEMMA 2.17. *The functor*

$$\mathrm{Mod}_R \to \mathrm{Desc}_f(\mathrm{Mod})$$

has a right adjoint given by sending

$$(N, \varphi) \mapsto \ker(N \xrightarrow{\varphi(n\otimes 1)-1\otimes n} S \otimes_R N)\ .$$

PROOF. We note that maps $(S \otimes_R M, \tau) \to (N, \varphi)$ in $\mathrm{Desc}_f(\mathrm{Mod})$ are given by maps $f : S \otimes_R M \to N$ of S-modules such that the diagram

$$\begin{tikzcd} S \otimes_R M \otimes_R S \arrow[r, "\tau"] \arrow[d, "f\otimes \mathrm{id}"] & S \otimes_R S \otimes_R M \arrow[d, "\mathrm{id}\otimes f"] \\ N \otimes_R S \arrow[r, "\varphi"] & S \otimes_R N \end{tikzcd}$$

of $S \otimes_R S$-linear maps commutes, where the upper map is the flip map. By the universal property of induction this is the same as a map of R-modules $M \to M$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}"] \arrow[d, "f\otimes 1"] & M \arrow[d, "1\otimes f"] \\ N \otimes_R S \arrow[r, "\varphi"] & S \otimes_R N \end{tikzcd}$$

But such a map is exactly a map into the kernel. $\square$

EXAMPLE 2.18 (Faithfully flat descent along $\mathbb{R} \to \mathbb{C}$). Let $f : \mathbb{R} \to \mathbb{C}$ be the canonical inclusion of fields. We claim $\mathrm{Desc}_f(\mathrm{Mod})$ is equivalent to the category of $\mathbb{C}$-vector spaces V together with an automorphism $\tau : V \to V$ which is $\mathbb{C}$-antilinear (meaning $\tau(zv) = \overline{z}\tau(v)$), and satisfies $\tau^2 = \mathrm{id}$.

Indeed, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to a product $\mathbb{C} \times \mathbb{C}$, along the map $a \otimes b \mapsto (a \cdot b, a \cdot \overline{b})$. So a $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$-module isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} V$ is determined by its basechanges along the two maps $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$. The first basechange takes both vector spaces to V, the second takes $V \otimes_{\mathbb{R}} \mathbb{C} \mapsto V$, and $\mathbb{C} \otimes_{\mathbb{R}} V \mapsto \overline{V}$. So a $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$-linear isomorphism $\varphi : V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} V$ can be identified with a pair of $\mathbb{C}$-linear isomorphisms $\varphi_0 : V \to V$, $\varphi_1 : V \to \overline{V}$.
Now we analyze the cocycle condition: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as $\mathbb{C}^{\times 4}$ by iterating the above decomposition, with the map taking $a \otimes b \otimes c \mapsto (abc, ab\overline{c}, a\overline{b}c, a\overline{b}\overline{c})$. Basechanging the diagram

$$\begin{tikzcd} V \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, "\varphi\otimes\mathrm{id}"] \arrow[dr, "\varphi_{(1)}\otimes\mathrm{id}\otimes\varphi_{(2)}"'] & \mathbb{C} \otimes_{\mathbb{R}} V \otimes_{\mathbb{R}} \mathbb{C} \arrow[d, "\mathrm{id}\otimes\varphi"] \\ & \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} V \end{tikzcd}$$

along the four projection maps leads to the four diagrams

$$\begin{tikzcd} V \arrow[r, "\varphi_0"] \arrow[dr, "\varphi_0"'] & V \arrow[d, "\varphi_0"] \\ & V \end{tikzcd} \quad \begin{tikzcd} V \arrow[r, "\varphi_0"] \arrow[dr, "\varphi_1"'] & V \arrow[d, "\varphi_1"] \\ & \overline{V} \end{tikzcd} \quad \begin{tikzcd} V \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_0"'] & \overline{V} \arrow[d, "\overline{\varphi_1}"] \\ & V \end{tikzcd} \quad \begin{tikzcd} V \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_0"'] & \overline{V} \arrow[d, "\overline{\varphi_0}"] \\ & \overline{V} \end{tikzcd}$$

of which the first tells us that $\varphi_0 = \mathrm{id}$ (since we know it is an isomorphism), the second and fourth are then trivial, and the third tells us that $\overline{\varphi_1} \circ \varphi_1 = \mathrm{id}$. Equivalently, we can view φ_1 as complex-antilinear map $\tau : V \to V$, which turns this condition into $\tau^2 = \mathrm{id}$. We also see that under this equivalence, the functor $\mathrm{Desc}_f(\mathrm{Mod}) \to \mathrm{Mod}_{\mathbb{R}}$ takes V to the subspace of v with $\tau(v) = v$.

EXAMPLE 2.19 (Real forms of $\mathbb{C} \times \mathbb{C}$). The equivalence between $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces with complex-antilinear involution given by the previous example is compatible with tensor products. So it also gives rise to an equivalence between $\mathbb{R}$-algebras and $\mathbb{C}$-algebras with complex-antilinear involution (which is an algebra map). We may use this to classify all $\mathbb{R}$-algebras A with $A \otimes_{\mathbb{R}} \mathbb{C}$ a given $\mathbb{C}$-algebra. For example, how many $\mathbb{R}$-algebras base-change to $\mathbb{C} \times \mathbb{C}$?
An anti-linear algebra involution τ on $\mathbb{C} \times \mathbb{C}$ needs to take the unit $(1,1)$ to iself, and since $(1,1)$ and $(1,0)$ are a $\mathbb{C}$-basis, it is determined by $\tau(1,0)$. This also has to be an idempotent, and so we have either $\tau(1,0) = (1,0)$ or $\tau(1,0) = (0,1)$. In the first case, the map generally takes $(a,b) \mapsto (\overline{a}, \overline{b})$, and thus the $\mathbb{R}$-algebra given by descent is $\mathbb{R} \times \mathbb{R}$. In the second case, the map takes $(a,b) \mapsto (\overline{b}, \overline{a})$, and we check that the subset of v with $\tau(v) = v$ is given by the set $(a, \overline{a})$ for $a \in \mathbb{C}$ and thus as an algebra isomorphic to $\mathbb{C}$. Thus we see that every $\mathbb{R}$-algebra A with $A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ is either isomorphic to $\mathbb{R} \times \mathbb{R}$ or $\mathbb{C}$.

EXAMPLE 2.20 (Real forms of $\mathbb{G}_m$). As a more involved example of the above classification, let us ask which $\mathbb{R}$-algebras A have $A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x^{\pm 1}]$. Geometrically, we are asking which schemes over $\mathrm{Spec}(\mathbb{R})$ pull back to $(\mathbb{G}_m)_{\mathbb{C}}$ over $\mathrm{Spec}(\mathbb{C})$.
A complex-antilinear automorphism τ of $\mathbb{C}[x^{\pm 1}]$ is determined by where it takes x, and it has to take it to a unit λx^k (all units are of this form). For τ to be an involution, it is necessary that $k = \pm 1$.
In the $k = 1$ case, τ^2 takes $x \mapsto \lambda \cdot \overline{\lambda} x$, so we need $|\lambda|^2 = 1$. Letting μ be a square root of λ, $\mu x \mapsto \overline{\mu} \lambda x = \mu x$, and so by replacing our generator x with μx, we may

assume $\lambda = 1$. Then $\tau(ax^n) = \overline{a}x^n$, and the $\mathbb{R}$-algebra determined by τ is given by $\mathbb{R}[x^{\pm 1}]$.
In the $k = -1$ case, τ^2 takes $x \mapsto \frac{\overline{\lambda}}{\lambda}x$, and so λ needs to be real. Since for any μ we have

$$\tau(\mu x) = \overline{\mu}\lambda x^{-1} = |\mu|^2\lambda(\mu x)^{-1},$$

replacing our generator x by μx, we may rescale λ by arbitrary positive real numbers. Thus we may assume $\lambda = 1$ or $\lambda = -1$.
In the $\lambda = 1$ case, $\tau(x) = x^{-1}$, and generally $\tau(ax^n) = \overline{a}x^{-n}$. The real subalgebra fixed by τ is therefore additively spanned by 1 and the elements $x^n + x^{-n}$ and $ix^n - ix^{-n}$ for $n > 0$. As algebra, it is generated by $u = \frac{x+x^{-1}}{2}$ and $v = \frac{ix-ix^{-1}}{2}$ (by a simple induction on degrees), and we have

$$v^2 = \frac{-x^2 + 2 - x^{-2}}{4} = 1 - u^2.$$

So this algebra can be identified with $\mathbb{R}[u, v]/(u^2 + v^2 = 1)$.
In the $\lambda = -1$ case, $\tau(x) = -x^{-1}$, and generally $\tau(ax^n) = \overline{a}(-1)^n x^{-n}$. So the real subalgebra fixed by τ is additively spanned by $x^n + (-1)^n x^{-n}$ and $ix^n - (-1)^n x^{-n}$, and again generated as algebra by $u = \frac{x-x^{-1}}{2}$ and $v = \frac{ix+ix^{-1}}{2}$. We have

$$v^2 = \frac{-x^2 - 2 - x^{-2}}{4} = -1 - u^2,$$

and thus this algebra can be identified with $\mathbb{R}[u, v]/(u^2 + v^2 = -1)$.
Geometrically, all three possibilities ($\mathrm{Spec}(\mathbb{R}[x^{\pm 1}])$, $\mathrm{Spec}(\mathbb{R}[u, v]/(u^2 + v^2 - 1))$, $\mathrm{Spec}(\mathbb{R}[u, v]/(u^2 + v^2 + 1))$) are schemes with complex points in bijection to $\mathbb{C}^\times$. They differ by how complex conjugation acts on the set of complex points: In the first case, it acts by $z \in \mathbb{C}^\times \mapsto \overline{z}$, fixing the subset $\mathbb{R}^\times$ of real points. In the second case, it acts as $z \in \mathbb{C}^\times \mapsto \overline{z}^{-1}$, fixing the subset $|z|^2 = 1$ (therefore, in the second case, the real points geometrically form a circle). Finally, in the third case, it acts as $z \in \mathbb{C}^\times \mapsto -\overline{z}^{-1}$, fixing no points, corresponding to the fact that there are no real points. One way to visualize schemes X over $\mathbb{R}$, related to this descent perspective, is as their set of complex points $X(\mathbb{C})$, "folded up" along the action of complex conjugation. In the three cases above, this leads to a "pointed closed half plane", a "pointed closed disk", and an "open Möbius strip", with boundaries corresponding to the real points. (This last object occurs as affine charts in a surprising real form of $\mathbb{P}^1_{\mathbb{C}}$, the "twistor $\mathbb{P}^1$")
Had we instead descended $\mathbb{C}[x]$ (i.e. $\mathbb{A}^1$), only the first case would occur, and all resulting real forms would be of the form $\mathbb{A}^1$, geometrically looking like a closed half plane.

Proof of Theorem 2.15. Now that we have an adjunction between

$$\mathrm{Mod}_R \underset{}{\overset{\otimes_R S}{\rightleftarrows}} \mathrm{Desc}_f(\mathrm{Mod})$$

in order to prove Theorem 2.15 it is enough to check that unit and counit of this adjunction are isomorphisms. Thus we have to check two things:

(1) For any descent datum (N, φ) the induced map

$$S \otimes_R \ker(N \xrightarrow{\varphi(n\otimes 1) - 1\otimes n} S \otimes_R N) \to N$$

is an isomorphism. A priori we need to check this as a morphism in the descent category, but by Lemma 2.16 it is enough to check this on underlying modules.

(2) For any R-module M the morphism

$$M \to \ker(S \otimes_R M \xrightarrow{\varphi(n\otimes 1)-1\otimes n} S \otimes_R S \otimes_R M)$$

is an isomorphism.

We claim that the second condition is implied by the first using the fact that the functor $\mathrm{Mod}_R \to \mathrm{Desc}_f(\mathrm{Mod})$ is conservative, i.e. detects isomorphism. In general we claim that for an adjunction with left adjoint

$$L : \mathcal{C} \to \mathcal{D}$$

that is conservative we get that the unit $c \to RLc$ is an isomorphism if the counit $LRd \to d$ is an isomorphism for every $d \in D$. Indeed, we can check whether $c \to RLc$ is an isomorphism by applying L to it, so that we get a morphism $Lc \to LRLc$. But this morphism sits in a commutative diagram

$$\begin{tikzcd} Lc \arrow[r, "\mathrm{id}"] & LRLc \arrow[d] \\ & Lc \end{tikzcd}$$

where the right vertical morphism is the counit of Lc which is an isomorphism by assumption.

The upshot of the previous discussion is that it is enough to check that the first condition above holds true. Now we observe that if we have any commutative diagram of rings

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"] & R' \arrow[d, "f'"] \\ S \arrow[r] & S' \end{tikzcd}$$

then we get an induced diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ \mathrm{Desc}_f \arrow[r] & \mathrm{Desc}_{f'} \end{tikzcd} .$$

If $R \to R'$ is flat then so is $S \to S'$ and under the lower horizontal morphism the map

$$S \otimes_R \ker(N \xrightarrow{\varphi(n\otimes 1)-1\otimes n} S \otimes_R N) \to N$$

gets mapped to the corresponding map for the induced descent object in $\mathrm{Desc}_{f'}$. Here flateness enters through the fact that the lower base-change otherwise does not necessarily preserve the kernel. If we moreover assume that $R \to R'$ is faithfully flat, then so is $S \to S'$ and thus the functor

$$\mathrm{Desc}_f \to \mathrm{Desc}_{f'}$$

is conservative. Thus in order to check that the adjunction counit for the first adjunction is an equivalence, we can check that this is the case for the second.

To do this we let R' be S so that we can replace the map $R \to S$ by the map $S \to S \otimes_R S$. But this second map has a retract. Thus we may without loss of generality assume that our first map $R \to S$ had a retract and thus reduced the result about faithfully flat descent to showing that for a map $f : R \to S$ with a retract $r : S \to R$ the functor

$$- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Desc}_f$$

is an equivalence of categories.

Now we would like to argue that this is indeed the case by establishing an inverse functor (which eventually has to agree with the other inverse functor). The inverse functor is given by

$$\mathrm{Desc}_f \xrightarrow{\mathrm{fgt}} \mathrm{Mod}_S \xrightarrow{r^*} \mathrm{Mod}_R$$

where r^* is base-change along the retraction. Now is is clear that the composition

$$\mathrm{Mod}_R \to \mathrm{Desc}_f \to \mathrm{Mod}_R$$

is equivalent to the identity, since $r \circ f \simeq \mathrm{id}$ and we are left with showing that the other composition

$$\mathrm{Desc}_f \to \mathrm{Mod}_R \to \mathrm{Desc}_f$$

is equivalent to the identity as well. To see this we need to prove that any descent datum (N, φ) is equivalent to the new descent datum $((fr)^*N, \tau)$. To see this we consider the map

$$r' : S \otimes_R S \to S$$

given by $s \otimes s' \mapsto r(s) \cdot s'$. Then for any S-module N we have

$$\begin{aligned} (r')^*(N \otimes_R S) &\cong (r')^*(i_1)^*(N) \cong (fr)^*N \qquad \text{and} \\ (r')^*(R \otimes_S N) &\cong (r')^*(i_2)^*N \cong N \end{aligned}$$

so that the isomorphism $\varphi : N \otimes_R S \to R \otimes_R N$ induces after basechange along r' an isomorphism

$$\theta : (fr)^*N \to N$$

of S-modules. We need to check that this isomorphism is compatible with the descent data, that is that the diagram

$$\begin{tikzcd} (fr)^*N \otimes_R S \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[d, "\tau"] & N \otimes_R S \arrow[d, "\varphi"] \\ S \otimes_R (fr)^*N \arrow[r, "\mathrm{id}\otimes\theta"] & S \otimes_R N \end{tikzcd} \tag{14}$$

commutes. To this end consider the commutative square of $S \otimes_R S \otimes_R S$-modules

$$\begin{tikzcd} N \otimes_R S \otimes_R S \arrow[r, "\varphi\otimes\mathrm{id}"] \arrow[dr, "\varphi_{(1)}\otimes\mathrm{id}\otimes\varphi_{(2)}"'] & S \otimes_R N \otimes_R S \arrow[d, "\mathrm{id}\otimes\varphi"] \\ & S \otimes_R S \otimes_R N \end{tikzcd}$$

which commutes by the cocycle identity. Now we base-change this diagram along

$$r' \otimes \mathrm{id} : S \otimes_R S \otimes_R S \to S \otimes_R S \qquad (s \otimes s' \otimes s'' \mapsto r(s) \cdot (s' \otimes s''))$$

and get a diagram

$$\begin{tikzcd} (fr)^*N \otimes_R S \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[dr, "(\mathrm{id}\otimes\theta)\circ\tau"'] & N \otimes_R S \arrow[d, "\varphi"] \\ & S \otimes_R N \end{tikzcd}$$

where we identity the diagonal arrow by explicitly noting that the map $\varphi_{(1)} \otimes \mathrm{id} \otimes \varphi_{(2)}$ is the base-change along $S \otimes_R S \to S \otimes_R S \otimes_R S$ given by $s \otimes t \mapsto s \otimes 1 \otimes t$ and using that the composite $(r' \otimes \mathrm{id})$ composed with this map equals $i_2 \circ r'$. This show that diagram 14 commutes and finishes the proof. □

Definition 2.21. *A morphism $f : Y \to X$ of schemes is a* fpqc cover *if it is flat, surjective (i.e. faithfully flat) and additionally every quasi-compact open subset of X is the image of a quasi-compact open subset of Y.*

Example 2.22. If $Y = \mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R)$ then $f : Y \to X$ is an fpqc cover iff the induced map $R \to S$ is faithfully flat. One direction is obvious, for the other one note that the morphism $Y \to X$ is quasi-compact, that is preimages of quasi-compact opens are quasi-compact (see Example 3.8). Hence we can simply take the preimage to verify the second condition.
More generally this argument shows that quasi-compact and faithfully flat morphisms are fpqc covers, that is also the reason for the name: fp = fidèlement plat (french for faithfully flat) and qc = quasi-compact. This is slightly confusing though, since the converse doesn't quite hold.

Example 2.23. Assume that $X = \mathrm{Spec}(R)$ and $f : Y \to X$ is a fpqc cover. Then we find $U \subseteq Y$ quasi-compact with $f(U) = X$. We can cover U by finitely many affine opens $\mathrm{Spec}(R_i) = U_i \subseteq U$ for $i = 1, ..., n$. Then we have a diagram

$$\begin{tikzcd} \coprod_{i=1}^n \mathrm{Spec}(R_i) \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

Thus we see that fpqc morphism are those which can be 'covered' by faithfully flat maps of rings.

Let $f : Y \to X$ be a fpqc cover of schemes. We define a category

$$\mathrm{Desc}_f(\mathrm{QCoh}) := \{\mathcal{M} \in \mathrm{QCoh}(Y), \varphi : \pi_1^*\mathcal{M} \xrightarrow{\sim} \pi_2^*\mathcal{M} \in \mathrm{QCoh}(Y \times_X Y) \mid \pi_{23}^*\varphi \circ \pi_{12}^*\varphi = \pi_{13}^*\varphi\}$$

where the latter equality holds in $\mathrm{QCoh}(Y \times_X Y \times_X Y)$.

Theorem 2.24. *Let $f : Y \to X$ be a fpqc morphism. Then the functor*

$$\mathrm{QCoh}(X) \to \mathrm{Desc}_f(\mathrm{QCoh})$$

is an equivalence of categories.

Proof. Omitted. □

3. Smooth morphisms

In this section we want to talk about smooth schemes over some base $\mathrm{Spec}(k)$. It will turn out that smoothness is really a notion for the morphism $X \to \mathrm{Spec}(k)$ and can be generalized to arbitrary base schemes S in place of $\mathrm{Spec}(k)$.
Recall from the introduction that tangent vectors and tangent spaces are related to morphisms

$$\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to X$$

and note that in $k[x]/x^2$ the element x squares to zero. We will now try to capture this behaviour geometrically:

Definition 3.1. *A closed immersion $i : S_0 \to S$ of schemes is called a first order thickening (or n-th order thickening) if the corresponding ideal sheaf $\mathcal{I} = \ker(i^\sharp : \mathcal{O}_S \to i_*\mathcal{O}_{S_0})$ satisfies $\mathcal{I}^2 = 0$ (or $\mathcal{I}^{n+1} = 0$).*

Examples of first order thickenings include the extension $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ as well as non-split extensions like $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}/p^2)$.

Lemma 3.2. *Every n-th order thickening $i : S_0 \to S$ is the composition of n first order thickenings*

$$S_0 \to S_1 \to \ldots \to S_n = S$$

Proof. Consider the ideal sheaves

$$\mathcal{I} = \mathcal{I}_0 \supset \mathcal{I}^2 = \mathcal{I}_1 \supset \ldots. \supset \mathcal{I}^{n+1} = \mathcal{I}_n = 0$$

and take the associated closed subschemes. Then we have that for an inclusion $S_i \to S_{i+1}$ the corresponding sheaf squares to zero since $\mathcal{I}_i^2 \subseteq \mathcal{I}_{i+1}$. □

Lemma 3.3. *For a n-th order thickening $S_0 \to S$ the induced map on underlying topological spaces is a homeomophism.*

Proof. The claim can be checked locally on S, since a continuous map $g : Z \to Z'$ of topological spaces is a homeomorphism precisely if there exists an open cover U_i of Z such that all induced morphisms $g^{-1}(U_i) \to U_i$ are homeomorphisms. Thus we may assume that S is affine. In this case we have the situation $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ where I is a nilpotent ideal. But then the induced map clearly is a homeomorphism. □

Definition 3.4. *A morphism $f : X \to S$ of schemes is called* formally smooth *if for every commutative diagram*

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

in which i is a first order thickening of **affine** *schemes there exists a lift*

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

rendering everything commutative. The morphism is called formally étale *if there exists a unique such lift and* formally unramified *if there exists at most one such lift.*

Clearly we have

$$\text{formally étale} \Leftrightarrow (\text{formally smooth} + \text{formally unramified})$$

Also note that by the lemma we could in the definition replace first order thickenings by n-th order thickenings since we can paste lifting squares. We warn the reader that the affinness for the test schemes cannot be dropped for formal smoothesness, but it can be dropped for unramifiedness and étaleness.

EXAMPLE 3.5. Every open immersion $U \to S$ is formally étale. To see this we consider the square

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "i"] & U \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

and note that a morphism $h : T \to S$ factors over U if and only if $h(|T|) \subseteq |U|$. In this case the lift is unique. The claim now follows since the map of underlying spaces $|T_0| \to |T|$ is a homeomorphism.

EXAMPLE 3.6. Every closed immersion $Z \to S$ is formally unramified. To see this note that for any scheme T the map

$$\mathrm{Hom}_{\mathrm{Sch}}(T, Z) \to \mathrm{Hom}_{\mathrm{Sch}}(T, S)$$

is injective, i.e. it is a property of a morphism $T \to S$ to factor through Z.
In general closed immersions are not formally smooth, e.g. consider a first order thickening $S_0 \to S$. Then for any square

$$\begin{tikzcd} S_0 \arrow[r, "\mathrm{id}"] \arrow[d, "i"] & S_0 \arrow[d, "i"] \\ S \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & S \end{tikzcd}$$

the dashed arrow would be an inverse to i. Thus first order thickenings are formally smooth iff they are formally étale iff they are isomorphisms.

EXAMPLE 3.7. For any commutative ring k the map $\mathbb{A}^n_k \to \mathrm{Spec}(k)$ is formally smooth, but not formally étale. To see this we have to consider a lifting problem

$$\begin{tikzcd} R/I & k[X_1, ..., X_n] \arrow[l, "g"'] \arrow[dl, dashed] \\ R \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

which can be solved by choosing preimages of the $g(X_i)$ under $R \to R/I$. Clearly those preimages are not unique.

EXAMPLE 3.8. Consider the map $\mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(\mathbb{F}_p)$ for a finite field $\mathbb{F}_q = \mathbb{F}_{p^n}$. We claim that it is formally étale. To see this we have to solve the lifing problem

$$\begin{tikzcd} R/I & \mathbb{F}_q \arrow[l, "g"'] \arrow[dl, dashed, "g'"] \\ R \arrow[u, "\pi"] & \mathbb{F}_p \arrow[l] \arrow[u] \end{tikzcd}$$

Let $x \in \mathbb{F}_q$ and note that $x = y^p$ for a unique element $y \in \mathbb{F}_q$. For any r with $\pi(r) = g(y)$ we have that

$$\pi(r^p) = \pi(r)^p = g(y)^p = g(x)$$

Moreover we claim that r^p is independent of the choice of r. Namely if r' is another element with $\pi(r') = g(y)$ then the difference $r - r'$ squares to zero and thus

$$r^p = r^p + (r' - r)^p = r'^p \ .$$

Now we can define

$$g'(x) := r^p \ .$$

It is easy to see that this is indeed a ringhomomorphism and solves the lifting problem. It is also unique.

Lemma 3.9. *The class of formally smooth/étale/unramified maps are closed under composition and pullback.*

Proof. By the very definition of lifting properties. □

Lemma 3.10. *The property that $f : X \to S$ is formally étale/umramified is local in the source, that is if there is an open cover $U_i \subseteq X$ with $i \in I$ such that all the maps*

$$f|_{U_i} : U_i \to S$$

are formally smooth, then f is formally smooth.

Proof. Since the classes are closed under composition by Lemma 3.9 and open immersions are formally étale, one direction is obvious.
Assume conversely that $f|_{U_i} : U_i \to S$ are formally unramified and we are given a diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

and two lifts $f, g : T \to X$. We set $T_i = f^{-1}(U_i) = g^{-1}(U_i)$, the equality follows since the underlying maps of topological spaces of f and g agree, as $T_0 \to T$ is a homeomorphism. Then $T_{0,i} \to T_i$ is a first order thickening, but not necessarily affine. Nevertheless we can conclude that $f = g$ on T_i, so that $f = g$ since the T_i cover T.
For formal étaleness assume that we are given a cover $T_i \subseteq T$ and locally defined lifts

$$\begin{tikzcd} T_{0,i} \arrow[r] \arrow[d] & X \arrow[d] \\ T_i \arrow[r] & S \end{tikzcd}$$

Then we can glue those to a global lift, since they are locally unique. □

Remark 3.11. One of the big results that we will prove soon is that formaly smootheness is also local in the source, but this will require the technology introduced in the next sections (Káhler differentials).

Definition 3.12. *A morphism $f : X \to S$ is locally of finite type (resp. finite presentation) if one of the following equivalent conditions hold:*

(1) *For any affine open* $U = \mathrm{Spec}(R) \subseteq X$ *mapping to* $\mathrm{Spec}(R') \subseteq S$ *the algebra* R *is a finitely generated* R' *algebra (resp. finitely presented).*
(2) *There exists an affine cover* $\mathrm{Spec}(R_i) = U_i$ *of* X *mapping to* $\mathrm{Spec}(R_i') = V_i \subseteq S$ *such that all* R_i *are finite generated (resp. presented)* R_i'*-algebras.*

Note that the difference between finite type and locally of finite type is that in the former case the morphism is additionally required to be quasi-compact.

Definition 3.13. *A morphism* $f : X \to S$ *is smooth (resp. étale, resp. unramified) if it is formally smooth (resp. formally étale, resp. formally unramified) and* f *is locally of finite presentation (resp. locally of finite presentation, resp. locally of finite type).*

Example 3.14. Open immersions are étale, $\mathbb{A}^n_k \to \mathrm{Spec}(k)$ is smooth, the map $\mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(\mathbb{F}_p)$ is étale.

4. Kähler differentials

Consider for any ring k the thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ with the map given by sending ϵ to 0. Note that $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ canonically a k-scheme.

Definition 4.1. *For any* k*-point* $x : \mathrm{Spec}(k) \to X$ *(over* $\mathrm{Spec}(k)$*) we define the tangent space* T_xX *to be the set of maps of* k*-schemes* $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ *extending* x*, i. the set of commutative squares*

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k[\epsilon]/\epsilon^2) \arrow[r] \arrow[ur] & \mathrm{Spec}(k) \end{tikzcd}.$$

Note that we always have a 'trivial' tangent vector induced by the map $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to \mathrm{Spec}(k) \xrightarrow{x} X$. Thus the map $X \to \mathrm{Spec}(k)$ being unramified in particular implies that the tangent space at every point consists only of one point. Also note that the definition of the tangent space only depends on an open neighborhood of x.

Definition 4.2. *For any* k*-algebra* A *and any* A*-module* M *a map* $\delta : A \to M$ *is a* k*-linear derivation if it is* k*-linear, additive and*

$$\delta(ab) = a\delta(b) + \delta(a)b \qquad \textit{(Leibniz rule)}$$

Remark 4.3. Note that if $\delta : A \to M$ is additive and satisfies the Leibniz rule, then being k-linear is equivalent to $\delta(\lambda) = 0$ for $\lambda \in k$: if this is the case then

$$\delta(\lambda a) = a\delta(\lambda) + \delta(a)\lambda = \delta(a)\lambda.$$

If conversely δ is k-linear then

$$\delta(\lambda) = \delta(\lambda \cdot 1) = \lambda\delta(1 \cdot 1) = 2\lambda\delta(1) = 2\delta(\lambda).$$

which implies $\delta(\lambda) = 0$.

We now want to identity T_xX for some $x \in X(k)$. We can without loss of generality assume that $X = \mathrm{Spec}(A)$.

Proposition 4.4. *The set of tangent vectors in* T_xX *is in 1-1-correspondence to* k*-linear derivations* $A \to k$*, where* k *is an* A*-module through the map* $x : A \to k$*.*

PROOF. We have to understand k-linear maps

$$A \to k[\epsilon]/\epsilon^2$$

over k, where the map $A \to k$ is given by x, such a map is given by a pair of k-linear maps $A \to k$ and $A \to k\epsilon$ where the first is determined by x. Thus the latter is of the form

$$a \mapsto \delta(a) \cdot \epsilon \ .$$

Thus we see that the tangent space is $1-1$ to k-linear maps $\delta : A \to k$ with the property that the induced map

$$h : A \to k[\epsilon]/\epsilon^2 \qquad a \mapsto x(a) + \delta(a)\epsilon$$

is a ring homomorphism. But we see that

$$h(ab) = x(ab) + \delta(ab)\epsilon = x(a)x(b) + \delta(ab)$$

and

$$h(a)h(b) = (x(a) + \delta(a)\epsilon) \cdot (x(b) + \delta(b)\epsilon) = x(a)x(b) + (x(a)b + x(b)a)\epsilon$$

which shows that δ has to be a derivation and finishes the proof. □

Note that the set of k-linear derivations $\mathrm{Der}_k(A, M)$ is an A-module by pointwise addition and scalar multiplication, so that we see that the tangent space T_xX is indeed an A-module in a canonical way (and thus in particular also a k-module).

PROPOSITION 4.5. *For any k-algebra A there is an initial k-linear derivation*

$$d : A \to \Omega^1_{A/k}$$

This is for any other derivation $d' : A \to M$ there is a unique k-linear morphism $f : \Omega^1_{A/k} \to M$ such that $fd = d'$.

The module $\Omega^1_{A/k}$ is called the module of Kähler differentials.

PROOF. We define $\Omega^1_{A/k}$ as the A-module freely generated by symbols da for $a \in A$ subject to the relations

$$d(ab) = ad(b) + bd(a) \qquad d(k) = 0 \qquad d(a+b) = da + db \ .$$

Then the map

$$A \to \Omega^1_{A/k}$$

is a k-linear derivation and it is clear that $\Omega^1_{A/k}$ is initial. □

COROLLARY 4.6. *We have an isomophism of k-modules*

$$T_xX = \mathrm{Der}_k(A, k) = \mathrm{Hom}_A(\Omega^1_{A/k}, k) = \mathrm{Hom}_k(\Omega^1_{A/k} \otimes_{k,x} k, k) = (\Omega^1_{A/k} \otimes_{k,x} k)^\vee \ .$$

EXAMPLE 4.7. We have $\Omega^1_{k/k} = 0$ since k-linear derivations $k \to M$ are identitically zero.

EXAMPLE 4.8. For $A = k[t]$ we have that

$$\Omega^1_{k[t]/k} = k[t] \cdot dt$$

is a free module on a single generator (called dt), which is seen by verifying the universal property: a k-linear derivation

$$\delta : k[t] \to M$$

is uniquely determined by $\delta(t)$ since we necessarily have

$$\delta(\sum a_i t^i) = \sum i t^{i-1} \delta(t) \ .$$

In other words: the map $\mathrm{Der}_k(A, M) \xrightarrow{\mathrm{ev}_t} M$ is a bijection. This show that the derivation $d : k[t] \to k[t] \cdot dt$ which sends p to $p' \cdot dt$ induces an bijection

$$\mathrm{Hom}_{k[t]}(k[t] \cdot dt, M) \to \mathrm{Der}_k(k[t], M) \qquad f \mapsto f \circ d \ .$$

as both are canonically isomorphic to M through evaluation at dt respectively t. Alternatively one could also use the generators and relations description

$$\Omega^1_{k[t]/k} = \frac{k[t] \cdot \{dp \mid p \in k[t]\}}{d(pq) = pdq + qdp, d\lambda = 0, d(p+q) = dp + dq}$$

to see that this is indeed generated by dt without any relations.

We have that

$$\Omega^1_{k[t_1,...,t_n]/k} = k[t_1, .., .t_n]\{dt_1, ..., dt_n\}$$

The proof is similar to the previous one, derivations $k[t_1, ..., t_n]$ are determined by their values on $t_1, ..., t_n$ without any relation between them. Alternatively since $\Omega^1_{k[t_1,...,t_n]/k} = k[t_1, .., .t_n]\{dt_1, ..., dt_n\} = \oplus k[t_1, .., .t_n]\{dt_i\}$ this also follows from the next lemma:

Lemma 4.9. *For general k-algebras A and B we have a canonical isomorphism*

$$\Omega^1_{A \otimes_k B/k} = \Omega^1_{A/k} \otimes_k B \oplus A \otimes_k \Omega^1_{B/k} \ .$$

Proof. We analyse k-linear derivations $\delta : A \otimes_k B \to M$. Here M is a $A \otimes_k B$-module, which means and A and a B-module with commuting module structures both leading to the same k-module structures. Clearly δ is determined by

$$\delta_1 = \delta(- \otimes 1) : A \to M$$

and

$$\delta_2 = \delta(1 \otimes -) : B \to M$$

and since $A \to A \otimes_k B$ is a ringhomomorphism we get that δ_1 and by symmetry also δ_2 are k-linear derivations. Moreover these clearly determine δ since

$$\delta(a \otimes b) = \delta((a \otimes 1) \cdot (1 \otimes b)) = \delta_1(a)b + a\delta_2(b).$$

For any pair of k-linear derivations $\delta_1 : A \to M$ and $\delta_2 : B \to M$ we find that

$$\delta : A \otimes_k B \to M$$

defined by

$$\delta(a \otimes b) := \delta_1(a)b + a\delta_2(b)$$

and linear extension is a derivation since

$$\delta((a \otimes b)(a' \otimes b')) = \delta_1(aa')bb' + aa'\delta_2(bb') = \delta_1(a)a'bb' + a\delta_1(a')bb' + aa'\delta_2(b)b' + aa'b\delta_2(b') \ .$$

which equals

$$\delta(a \otimes b)a'b' + ab\delta(a' \otimes b') \ .$$

Together this shows that $\delta \mapsto (\delta_1, \delta_2)$ determines an isomorphism

$$\begin{aligned}
\mathrm{Der}_k(A \otimes_k B, M) &= \mathrm{Der}_k(A, M) \times \mathrm{Der}_k(B, M) \\
&= \mathrm{Hom}_A(\Omega^1_{A/k}, M) \times \mathrm{Hom}_B(\Omega^1_{B/k}, M) \\
&= \mathrm{Hom}_{A\otimes_k B}(\Omega^1_{A/k} \otimes_k B, M) \times \mathrm{Hom}_{A\otimes_k B}(A \otimes_k \Omega^1_{B/k}, M) \\
&= \mathrm{Hom}_{A\otimes_k B}(\Omega^1_{A/k} \otimes_k B \oplus A \otimes_k \Omega^1_{B/k})
\end{aligned}$$

which by naturality in M finishes the proof. $\square$

Lemma 4.10. *We have*

$$\Omega^1_{A\otimes_k B/B} = \Omega^1_{A/k} \otimes_k B$$

Proof. Follows from universal properties, similar to the last proof. $\square$

Proposition 4.11. *Assume we have maps $k \to A \to B$ of rings.*

(1) *The sequence*

$$\Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k} \to \Omega^1_{B/A} \to 0$$

induced by the 'canonical' maps of B-modules sending dx to dx is exact.

(2) *If $A \to B$ is surjective with kernel I then we have an exact sequence*

$$I/I^2 \xrightarrow{d\otimes 1} \Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k} \to 0\ .$$

Proof. 1) The surjectivity of the map $\Omega^1_{B/k} \to \Omega^1_{B/A}$ is clear since both are generated as B-modules by symbols of the form db. Moreover from the generators and relations descriptions we also see that

$$\Omega^1_{B/A} = \frac{\Omega^1_{B/k}}{B\{da \mid a \in A\}}$$

which also shows exactness in the middle.
2)
For the next claim note that because $B = A/I$ as A-modules we have

$$\Omega^1_{A/k} \otimes_A B = \frac{\Omega^1_{A/k}}{I\Omega^1_{A/k}}$$

and

$$(f_1, ..., f_n)/\Omega^1_{B/k} = \frac{\Omega^1_{A/k}}{A\{d(i) \mid i \in I\} + I\Omega^1_{A/k}}$$

From this we get the desired result, since the kernel of the map is generated as an A-module by $d(i)$. TODO: some details missing $\square$

Example 4.12. Consider $B = k[t]/f$ for some polynomial $f \in k[t]$. With $A = k[t]$ we find that

$$\Omega^1_{B/k} = (k[t]/f \cdot dt)/f'dt = (k[t]/(f, f')) = B/f' \cdot dt.$$

The conormal sequence takes the form

$$(f)/(f^2) \xrightarrow{d} Bdt \to B/f'dt \to 0\ .$$

For $f = x^2$ we see that the first map is not injective, as for example $[x^3]$ lies in the kernel. More generally for $B = k[t_1, ..., t_n]/f_1, ..., f_r$ we find that

$$\Omega^1_{B/k} = B\{dt_1, ..., dt_n\}/df_1, ..., df_n$$

as B-modules.

EXAMPLE 4.13. Let k be a field and consider the spectrum $X = \mathrm{Spec}(k[x,y]/xy) = \mathrm{Spec}(A)$, i.e. the coordinate axis. We want to calculate the tangent space $T_{\alpha,\beta}$ for k-valued points α, β. We find that

$$\Omega^1_{A/k} = A\{dx, dy\}/(xdy + ydx)\ .$$

If $\alpha \in k^\times$ and $\beta = 0$ then

$$\begin{aligned} T_{\alpha,0} &= \left(\Omega^1_{A/k} \otimes_{A,\phi_{\alpha,\beta}} k\right)^\vee \\ &= (k\{dx, dy\}/\alpha dy)^\vee \\ &= (k\{dy\})^\vee = k \cdot \frac{\partial}{\partial y} \end{aligned}$$

where $\partial/\partial y$ is the derivation $A \to k$ that takes the partial derivative by y and then evaluates at $(\alpha, 0)$. Similarly we find for $(0, \beta)$ with $\beta \in k^\times$ that

$$T_{(\beta,0)} = k \cdot \frac{\partial}{\partial x}\ .$$

Finally for the point $(0,0)$ we have that

$$T_{(0,0)} = (k\{dx, dy\})^\vee = k\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$$

so that this tangent space is 2-dimensional (and agrees with the tangent space of the ambient plane).

Now we want to generalize the relation between derivations and tangent vectors to arbitrary first order thickenings. Thus assume that we are given a diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}(B/I) = T_0 & \longrightarrow & X = \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) = T & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

or equivalently a diagram of rings

$$\begin{array}{ccc} B/I & \xleftarrow[\bar{f}]{} & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & k \end{array}$$

We assume that $I^2 = 0$. In particular I is naturally a B/I-module, since the canonical B-module structure factors over B/I. Thus through the map $A \to B/I$ we can given I the structure of an A-module.

PROPOSITION 4.14. *The set of lifts in the above diagram $A \to B$ forms a torsor over the group of k-linear derivations $\delta : A \to I$. Concretely for a lift $f : A \to B$ and a k-linear derivation $\delta : A \to I$ we get a new lift as $f + \delta : A \to B$.*

Proof. Clearly for $f+\delta$ the diagram also commutes. Moreover it is a ringhomomophism since

$$\begin{aligned}(f+\delta)(ab) &= f(ab)+\delta(ab)\\ &= f(a)f(b)+\delta(a)\overline{f}(b)+\overline{f}(a)\delta(b)\\ &= f(a)f(b)+\delta(a)f(b)+f(a)\delta(b)\\ &= (f(a)+\delta(a))(f(b)+\delta(b)) = (f+\delta)(a)(f+\delta)(b)\ .\end{aligned}$$

Conversely we claim that for any pair of lifts f, f' the difference $f-f' : A \to I$ is a derivation (note that it lands in I by the fact that both lift $\overline{f}$). This is straightforward:

$$\begin{aligned}(f-f')(ab) &= f(ab)-f'(ab)\\ &= f(a)f(b)-f'(a)f'(b)\\ &= (f-f')(a)f(b)+f'(a)(f-f')(b)\\ &= (f-f')(a)\overline{f}(b)+\overline{f}(a)(f-f')(b)\ .\end{aligned}$$

□

Corollary 4.15. *A morphism* $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ *is formally unramified if and only if* $\Omega^1_{A/k} = 0$.

Proof. If $\Omega^1_{A/k} = 0$ then there are no non-trivial derivations $A \to I$ to any A-module I, in particular any pair of lifts as above have to agree. If conversely f is formally unramified then we consider for any A-module M the ring $B = A \oplus \epsilon M$ with 'square-zero'-multiplication, i.e.

$$(a+\epsilon m)(a'+\epsilon m') = aa' + am' + ma'\ .$$

This contains M as an ideal and we consider the square

$$\begin{array}{ccc} A & \xleftarrow[\mathrm{id}]{} & A \\ \uparrow & & \uparrow \\ A+\epsilon M & \longleftarrow & k \end{array}$$

There clearly exist lifts in this square (e.g. the inclusion $A \to A \oplus \epsilon M$) and any other lift differs by a k-linear derivation $\delta : A \to M$. We conclude that there are no such derivations and thus that $\Omega^1_{A/k} = 0$. □

Note that the last proof also shows that derivations $\delta : A \to M$ are the same as k-linear sections of the map $A + \epsilon M \to M$ which sends ϵm to 0.

Proposition 4.16. *Assume* $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ *is formally smooth. Then* $\Omega^1_{A/k}$ *is projective as an* A*-module.*
If f *is smooth then* $\Omega^1_{A/k}$ *is finitely generated projective, i.e. a vector bundle.*

Proof. Assume we are given any surjection of A-modules $M \to \Omega^1_{A/k}$ we need to produce a section $s : \Omega^1_{A/k} \to M$ or equivalently a k-linear derivation $\delta : A \to M$ such that the composition $A \to M \to \Omega^1_{A/k}$ is given by d. Equivalently we try to find a k-linear section of $A + \epsilon M \to A$ such that the induced section of $A + \epsilon\Omega^1_{A/k} \to A$ is given by $\mathrm{id} + \epsilon d$. This can equivalently be described as a lifting problem

$$\begin{tikzcd} A + \epsilon\Omega^1_{A/k} & A \arrow[l, "\mathrm{id}+\epsilon d"'] \\ A + \epsilon M \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Now the map $A + \epsilon M \to A + \epsilon\Omega^1_{A/k}$ is surjective and the kernel squares to zero. Thus such a lift exists.
For the last claim we simply observe that if $A = k[t_1, ..., t_n]/I$ then $\Omega^1_{A/k}$ is a quotient of $A\{dt_1, ..., dt_n\}$ which shows that it is finitely generated. $\square$

Proposition 4.17. *For morphisms $k \to A \to B$ with $A \to B$ formally smooth the sequence*

$$0 \to \Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k} \to \Omega^1_{B/A} \to 0$$

of Proposition 4.11 is split short exact.[2]

Proof. Using Proposition 4.11 it is enough to produce a retract of the map $\Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k}$. Using that the source is induced and the property of Kähler differentials this translates into a a k-linear derivation $r : B \to \Omega^1_{A/k} \otimes_A B$ such that the composition

$$A \to B \xrightarrow{r} \Omega^1_{A/k} \otimes_A B$$

is $d \otimes 1$. Equivalently we search a section of $B + \epsilon(\Omega^1_{A/k} \otimes_A B) \to B$ such that the induced map $A \to B + \epsilon(\Omega^1_{A/k} \otimes_A B)$ is the map $a \mapsto f(a) + \epsilon(d(a) \otimes 1)$. This translates into the lifting problem

$$\begin{tikzcd} B & & B \arrow[ll, "\mathrm{id}"'] \\ B + \epsilon(\Omega^1_{A/k} \otimes_A B) \arrow[u] & & A \arrow[ll] \arrow[u] \end{tikzcd}$$

Note that any section is automatically k-linear so that we find a lift and finish the proof. $\square$

Proposition 4.18. *For morphisms $k \to A \to B$ with $A \to B$ surjective with kernel I and $k \to B$ formally smooth the sequence I*

$$0 \to I/I^2 \xrightarrow{d} \Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k} \to 0\ .$$

of Proposition 4.11 is split short exact.

Proof. We want to construct a B-linear retraction of the first map. Unwinding what that means we have to find a derivation

$$\delta : A \to I/I^2$$

[2] Note that the target is projective, so being split short exact follows immediately from being short exact. The point here is that the first map is injective

such that the composition $I \to A \to I/I^2$ is the canonical map. We now use the square

$$\begin{tikzcd} A/I \arrow[d, leftarrow] & A/I \arrow[l, "\mathrm{id}"'] \arrow[d, leftarrow] \arrow[dl, dashed] & A \arrow[l] \arrow[dl, leftarrow] \\ A/I^2 & k \arrow[l] & \end{tikzcd}$$

and find a lift as indicated since $k \to A/I = B$ is formally smooth. Now we have two lifts in the outer diagram, since the projection $A \to A/I^2$ is also a lift. The dfifference gives the desired derivation as the kernel of $A/I^2 \to A/I$ is $I/I/I^2$. $\square$

We now want to define for every scheme X over some base S a quasi-coherent sheaf $\Omega^1_{X/S}$ which on affines is given by the module of Kähler differentials. Then we will translate the local statements that we had into global ones.

Proposition 4.19. *The construction $A \mapsto \Omega^1_{A/k}$ commutes with localizations, that is*

$$\Omega^1_{A[b^{-1}]/k} \cong \Omega^1_{A/k} \otimes_A A[b^{-1}]$$

and if A is $k[c^{-1}]$-algebra then

$$\Omega^1_{A/k[c^{-1}]} \cong \Omega^1_{A/k}$$

In particular we obtain for every scheme X over a scheme S a quasi-coherent sheaf $\Omega^1_{X/S}$ on X such that the value on affine opens $\mathrm{Spec}(A) \subseteq X$ *is given by the Kähler-differentials of A.*

Proof. Let M be a module over $A[b^{-1}]$ which we can equivalently think of an A-module on which b acts inertibly. We claim that the restriction along $A \to A[b^{-1}]$ induces an iso

$$\mathrm{Der}_k(A[b^{-1}], M) \to \mathrm{Der}_k(A, M) \ .$$

which shows the claim since the two sides are corepresented by the objects in question. To see this note that for a derivation $\delta : A[b^{-1}] \to M$ we necessarily have

$$0 = \delta(1) = \delta(bb^{-1}) = \delta(b)b^{-1} + b \cdot \delta(b^{-1})$$

thus

$$\delta(b^{-1}) = -b^{-2}\delta(b)$$

and therefore

$$\delta(ab^{-n}) = \delta(a)b^{-n} - nab^{-n-1}\delta(b) \ .$$

Thus the derivation is completely determined on A and for a derivation on A the formula defines a derivation on $A[b^{-1}]$. $\square$

Now from the statements of this section we immediately obtain the following non-affine versions:

Proposition 4.20. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes.*

(1) *There exists a canonical derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ which restricts to the universal derivation in the affine situation.* [3]

[3]Here such a map is called derivation if for all $U \subseteq X$ the induced map $\mathcal{O}_X(U) \to \Omega^1_{X/S}(U)$ is a derivation.

(2) *The sheaf* $\Omega^1_{X/S}$ *commutes with base change with respect to* S*, that is if we have a map* $S' \to S$ *and set* $X' := X \times_S S'$ *with* $g : X' \to X$ *then*

$$\Omega^1_{X'/S'} \cong g^*\Omega^1_{X/S} \ .$$

(3) *The sequence,*

$$f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact. Here the first map sends da *for* $a \in \mathcal{O}_Y$ *to* $df^\sharp(a)$*. If* f *is formally smooth, then the sequence above is exact on the left (more precisely, the first map is split injective).*

(4) *If* f *is a closed immersion with corresponding ideal sheaf* $\mathcal{I}$ *then the sequence*

$$f^*(\mathcal{I}) \xrightarrow{d} f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S} \to 0$$

is exact. If gf *is formally smooth, then our sequence is exact on the left and locally split.*

(5) *If* f *is smooth then* $\Omega^1_{X/Y}$ *is a vector bundle.*

(6) f *is formally unramifired if and only if* $\Omega^1_{X/Y} = 0$*.*

(7) *If* f *is formally étale then* $f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$ *is an isomorphism.*

PROOF. The key is to note that all the maps can be constructed locally as long as they are compatible with restriction. After constructing the maps, all the statements can be checked locally in which case they reduce to Lemma 4.10, Propositions 4.11, 4.16, 4.17, 4.18 and Corollary 4.15. For (6) we also use Lemma 3.10 and for (5) that if a map is smooth then it is locally smooth, which is clear by definition. The converse is also true and will be proved in the next section, but not needed here. The last part (7) follows by combining (3) and (6). $\square$

5. Formal smootheness is a local property

Our goal in this section is to prove an analogue of Lemma 3.10 for smooth morphisms. That is the following result:

THEOREM 5.1. *Assume that we have a morphism* $f : X \to S$ *and open covers* $U_i \subseteq X$ *and* $V_i \subseteq S$ *with* $f(U_i) \subseteq V_i$*. Then* f *is (formally) smooth if and only if each restriction* $f_i : U_i \to V_i$ *are (formally) smooth.*

We will mostly be intersted in the case where $V_i = S$. c

EXAMPLE 5.2. The scheme $\mathbb{P}^n_k$ is smooth over $\mathrm{Spec}(k)$ since there is an open cover of $\mathbb{P}^n_k$ given by opens isomorphic to $\mathbb{A}^n_k$ and we already know that $\mathbb{A}^n_k \to \mathrm{Spec}(k)$ is smooth.

So the reason that $\mathbb{P}^n_k$ is smooth is that it locally looks like $\mathbb{A}^n_k$. More generally we see that a scheme $X \to S$ is smooth if it is locally isomorphic to an open subset of $\mathbb{A}^n_S$. However it turns out that the converse does not quite hold for general smooth schemes, i.e. not every smooth scheme has this property. But it does hold up to passing to étale covers of open subsets of $\mathbb{A}^n_k$. More precisely we have the following local form of smooth schemes:

Corollary 5.3 (Uniformising Parameters). *Given a map of schemes $f : X \to S$. Then f is smooth if and only if for each $x \in X$ there is an open neighbourhood $x \in U \subseteq X$ and sections $x_1, ..., x_n \in \Gamma(U, \mathcal{O}_U)$ such that the map*

$$(x_1, ..., x_n) : U \to \mathbb{A}^n_S$$

is an étale map and that $\Omega^1_{X/S}|_U = \bigoplus_{i=1}^n \mathcal{O}|_U dx_i$.

Proof. One direction is easy: since étale maps are smooth and $\mathbb{A}^n_S$ is smooth we find that every scheme that has the prescribed property is smooth.
Assume conversely that $f : X \to S$ is smooth. We know that $\Omega^1_{X/S}$ is a vector bundle 4.20. We also know that $\Omega^1_{X/S}|_U$ for $U \subseteq X$ is generated by terms of the form dg for $g \in \mathcal{O}_U$ (e.g. take U to be affine). We claim that, after possibly further shrining U we can in fact find a basis of $\Omega^1_{X/S}|_U$ by elements of the form $dx_1, ..., dx_n$. This is a general claim: given a vector bundle $\mathcal{V}$ over a scheme S with a set of generators $G \subseteq \mathcal{V}(S)$ as an $\mathcal{O}_S$-module. Then locally around each point we can find an isomorphism

$$\mathcal{V}|_U \cong \bigoplus_{g \in G} \mathcal{O}_U \ .$$

This intermediate claim is an exercise (use Nakayama). Now we choose elments $x_1, ..., x_n \in \mathcal{O}(U)$ such that

$$\Omega^1_{U/S} = \Omega^1_{X/S}|_U = \bigoplus_{i=1}^n \mathcal{O}|_U \cdot dx_i$$

and consider the resulting commutative triangle

$$\begin{tikzcd} U \arrow[rr, "g=(x_1,...,x_n)"] \arrow[dr, "f"'] & & \mathbb{A}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

Clearly g is locally of finite presentation, since f ist. The induced map

$$g^*(\Omega^1_{\mathbb{A}^n_S/S}) \to \Omega^1_{U/S}$$

is an isomophism by construction. We get immediately from Proposition 4.20 (3) and (6) that g is formally unramified. Thus it remains to show that g is formally smooth, which follows from the next assertion. □

Proposition 5.4. *Assume we have a morphism $g : X \to Y$ of schemes over S where $X \to S$ is formally smooth then g is formally smooth if and only if the canonical map*

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is locally split injective.

Proof. One direction is immediate from Proposition 4.20 (3). For the other assertion assume that

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is locally split injective. SInce formal smootheness can be checked locally we may assume $S = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$, so that we have a map $g : A \to B$ of k-algebras and we may assume that the map

$$\Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k}$$

admits a retraction r. We need to solve the lifting problem

$$\begin{tikzcd} R/I & B \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & A \arrow[l, "u_0"] \arrow[u, "g"'] \end{tikzcd}$$

We can solve this problem if we replace A by k by the assumption that $X \to S$ is smooth. Thus we find a morphism $u : B \to R$ such that the upper triangle commutes but the lower triangle does not necessarily commute, but the diagram

$$\begin{tikzcd} R/I & B \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] \arrow[r] & k \arrow[u] \end{tikzcd}$$

commutes. We would like to modify u to another section $u' = u + \delta$ of the lower diagram, so that the upper diagram also commutes, i.e. $u'g = u_0$. Thus the desired δ has to be a k-linear derivation $\delta : B \to I$ such that we have the equality

$$\delta \circ g = (u' - u)g = u_0 - ug. \tag{15}$$

Note that the right hand side is a derivation $A \to I$ where I is an A-module through $u_0 : A \to I$. We denote this derivation as $\delta_0 : A \to I$ so that we are looking for a derivation $\delta : B \to I$ which restricts to $\delta_0 : A \to I$.
Our δ can be considered as a map

$$\Omega^1_{B/k} \to I$$

and (15) translates into the requirement that the composite

$$\Omega^1_{A/k} \to \Omega^1_{B/k} \to I$$

is the map associated with δ_0. Since I is a B-module the latter is equivalent to the requirement that the composite

$$\Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k} \to I$$

is the map induced by δ_0. Now the latter map has a retraction r, so that we can find such a derivation by composition with the retraction:

$$\Omega^1_{B/k} \xrightarrow{r} \Omega^1_{A/k} \otimes_A B \to I$$

which finishes the proof. □

Now we would like to prove Theorem 5.1. Note first that being of locally finite type is a local property, thus it suffices to prove the result for formally smooth morphisms (even though we will see that it is easier for smooth morphisms). In the course of the proof we will employ the theory of torsors and H^1 that we had introduced last term (cf Section 9).
In order to show that $X \to S$ is formally smooth we have to solve a lifting problem

$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d] & X \arrow[d] \\ T \arrow[r, "u"] & S \end{tikzcd}$$

We fix this lifing problem and in particular the maps u_0 and u for the rest of the proof. By assumption we can solve this lifting problem if we replace T be open affine subsets $U \subseteq T_0$ which under the map $|T_0| \to |X|$ are mapped to some open U_i. Then we get an induced lifting problem

$$\begin{array}{ccccc} U_0 & \longrightarrow & U_i & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & V_i & \longrightarrow & S \end{array}$$

which we can solve by the assumption that $U_i \to V_i$ is formally smooth. In other words: if we consider the sheaf $L \in \mathrm{Shv}(|T_0|, \mathrm{Set})$ defined by

$$U \subseteq T \mapsto \left\{ \text{lifts in } \begin{array}{ccc} U_0 & \xrightarrow{u_0} & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array} \right\}$$

then this locally admits sections. Our task is to show that it admits a global section.

Proposition 5.5. *The sheaf L is a torsor over the sheaf of groups*

$$\mathcal{G} = \underline{\mathrm{Hom}}_{\mathcal{O}_{T_0}}((u_0)^*\Omega^1_{X/S}, \mathcal{I}) \ .$$

Here $\mathcal{I}$ is the ideal defining T_0 in T which can be considered as a quasi-coherent sheaf on T_0 by the fact that we are an affine first order thickening.The inner hom $\underline{\mathrm{Hom}}_{\mathcal{O}_{T_0}}(\mathcal{F}, \mathcal{G})$ is the sheaf whose U-sections are given by

$$\underline{\mathrm{Hom}}_{\mathcal{O}_{T_0}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_{T_0}|U}(\mathcal{F}|_U, \mathcal{G}|_U) \ .$$

Proof. This is the global version of Proposition 4.14 in combination with the universal property of Kähler differentials. We will spell out the argument a bit more: We first analyse what maps

$$\tilde{\delta} : (u_0)^*\Omega^1_{X/S}|_U \to \mathcal{I}|_U$$

are for $U \subseteq T_0$. Let us denote the inclusion $U \to T_0$ by i. Then we have that

$$(u_0)^*\Omega^1_{X/S}|_U = i^* u_0^* \Omega^1_{X/S}$$

Thus a map $\tilde{\delta}$ is by adjunction equivalently given by a $\mathcal{O}_X$-linear map

$$\Omega^1_{X/S} \to (u_0)_* i_* \mathcal{I}$$

Now we claim that such morphisms are the same as derivations

$$\mathcal{O}_{X/S} \to (u_0)_* i_* \mathcal{I} \ .$$

To verify this claim we note that derivations as well as maps $\Omega^1_{X/S} \to (u_0)_* i_* \mathcal{I}$ are global sections of sheaves, there is a natural map connecting them (induced by the universal derivation) and that this map is locally an isomorphism.

On the other hand let us say what lifts in the upper $l \in L(U)$ are. First the underlying map of topological spaces $|l|$ is uniquely determined since $T_0 \to T$ is a homeomorphism. Thus we only need to understand the map

$$l^\sharp : \mathcal{O}_X \to l_* \mathcal{O}_T|_U = (u_0)_* i_* \mathcal{O}_T$$

where the last equality only makes sense after identitying the underlying topological spaces of T_0 and T.

Having given this translation it is clear that we can act with a derivation δ on a lift l by addition (all the well-definedness, commutativity of squares etc. can be checked locally where it reduces to the claims made in Proposition 4.14). Finally also the assertion about this being a torsor can be reduced to the local case: if $u_0 i(U)$ is contained in some affine open $\mathrm{Spec}(A) \subseteq X$ then everything literally reduces to the local situation. $\square$

In order to finish the proof of Theorem 5.1 we need to show that the $\mathcal{G}$-torsor L is trivial, equivalently admits a global section. We will do this by showing that all torsors over the specific sheaf of groups $\mathcal{G}$ are trivial, that is

$$H^1(T, \mathcal{G}) = 0$$

For this we need the following result:

PROPOSITION 5.6. *Given an affine scheme Z, $\mathcal{M}$ a quasi-coherent sheaf corresponding to a projective module and $\mathcal{N}$ a quasi-coherent sheaf. Then*

$$H^1(Z, \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{N})) = 0$$

PROOF. We write $\mathcal{M}$ as a retract of $\bigoplus_I \mathcal{O}_Z$ and thus find that $\underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{N})$ is a retract of

$$\prod_I \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{N}) = \prod_I \mathcal{N}$$

so that $H^1(Z, \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{N}))$ is a retract of

$$H^1\Big(Z, \prod_I \mathcal{N}\Big) .$$

and it suffices to show that the latter vanishes. Now we claim that the canonical map

$$H^1\Big(Z, \prod_I \mathcal{N}\Big) \to \prod_I H^1(Z, \mathcal{N})$$

is injective. To see this we note that $H^1\Big(Z, \prod_I \mathcal{N}\Big)$ is given by isomorphism classes of torsors over $\prod_I \mathcal{N}$. We claim that such a torsor is the same as a family of torsors $\mathcal{P}_i$ over $\mathcal{N}$ for each i such that for every point $z \in Z$ there exists a neighboorhoood in which all the torsors admit sections. To see this we simply note that every $\prod_I \mathcal{N}$ torsor leads to such a family by"base-change" and that conversly taking infinite products defines an inverse map. $\square$

PROOF OF THEOREM 5.1. Assume that $U_i \to V_i$ is smooth for some cover. Then we have that $\Omega^1_{U_i/V_i}$ is locally projective in the sense that is is locally projective on affines. But

$$\Omega^1_{U_i/V_i} = \Omega^1_{U_i/S} = \Omega^1_{X/S}|_{U_i}.$$

We conclude that $(u_0)^*\Omega^1_{X/S}|_U$ is projective for $U \subseteq T_0$ small enough. In particular we can find an affine open cover of T_0 so that the restriction is projective. Now if the maps $U_i \to V_i$ were smooth we would now that $(u_0)^*\Omega^1_{X/S}|_U$ is also finitely generated, i.e. a vector bundle, so that it corresponds to a projective module over $\Gamma(T_0)$. Without the finiteness assumption we need to use the highly non-trivial result of Raynaud and Gruson the projectivity of an R-module can be checked Zariski-locally. Then the discussion above finishes the proof. $\square$

6. Criteria for smootheness

PROPOSITION 6.1. *Assume we have a closed immersion $i : Z \to X$ of schemes over S where $X \to S$ is formally smooth, then $Z \to S$ is formally smooth if and only if the canonical map*

$$i^*\mathcal{I} \xrightarrow{d} i^*\Omega^1_{X/S}$$

is locally split injective.

PROOF. One direction is immediate from Proposition 4.20 (4). For the converse we assume that the map is locally split injective. Using Theorem 5.1 we can reduced to the affine situatution, i.e. $X = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(A/I)$ and $S = \mathrm{Spec}(k)$. The splitting translates into the condition that $I/I^2 \xrightarrow{d} \Omega^1_{A/k} \otimes_A A/I$ is split injective. We need to solve a lifting problem of the form

$$\begin{tikzcd} B/J & A/I \arrow[l] \arrow[ddl, dashed] \\ & A \arrow[u] \\ B \arrow[uu] & k \arrow[l] \arrow[u] \end{tikzcd}$$

with $J^2 = 0$. We can find a lift $u : A \to B$ starting at A by smootheness:

$$\begin{tikzcd} B/J & A \arrow[l, "u_0"'] \arrow[dl, dashed, "u"] \\ B \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

We need to modify u to a different splitting $u' = u - \delta$ for some derivation $\delta : A \to J$ such that u' vanishes on I, i.e. that $\delta|_I = u|_I$. Note that $u|_I$ lands automatically in J and vanishes in I^2.
To find a derivation δ is equivalent to finding a A/I-linear map

$$\Omega^1_{A/k} \otimes_A A/I \to J$$

which has the property that the composite $I/I^2 \to \Omega^1_{A/k} \otimes_A A/I \to J$ is given by . u. But this can be achieved using the retract. $\square$

Consider a closed subscheme

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr] & & \mathbb{A}^n_k \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

for k a ground ring such that $Z \to \mathbb{A}^n_k$ is locally of finite presentation. If we write $Z = V(I)$ for $I \subseteq k[X_1, ..., X_n]$ then this implies that I is a finitely generated ideal.

Set $B = k[X_1, ..., X_n]/I$. According to the last statement smoothness of Z over $\mathrm{Spec}(k)$ means that the B-linear map

$$I/I^2 \xrightarrow{d} \Omega^1_{k[X_1,..,X_n]/k} \otimes_{k[X_1,....,X_n]} B = B\{dx_1, ..., dx_n\}$$

is locally (in B) split injective. We want to make this more explicit now.

Let $z \in Z$ be a point and we want to investigate what it means that this is true around z. We consider the vector space $V := I/I^2 \otimes_B \kappa(x)$.[4] We choose a set of elements $f_1, ..., f_k \in I$ such that the images form a basis of V (this is possible since I is finitely generated). Then by Nakayama we have that in some open neighborhood $U \subseteq \mathbb{A}^n_k$ of z the elements

$$f_1, ..., f_k$$

form a generating set of the ideal I. Geometrically this means that we have

$$Z \cap U = V(f_1, ..., f_k) \cap U \ .$$

Now in order to verify that the map

$$d : I/I^2 \to B\{dx_1, ..., dx_n\}$$

is split injective we consider the B-linear composition

$$B\{f_1, ..., f_k\} \to I/I^2 \xrightarrow{d} B\{dx_1, ..., dx_n\}$$

which is given by the Jacobi matrix

$$\mathcal{J} = (\partial f_i/\partial x_j)_{i,j} \in M(n \times k, B).$$

In the neighborhood U the first map $B\{f_1, ..., f_k\} \to I/I^2$ is surjective and after base-change to $\kappa(x)$ an isomorphism. Thus if the second map is locally split injective, then the base-change of the Jacobi matrix to $\kappa(x)$

$$\mathcal{J}_x = \mathcal{J} \otimes_B \kappa(x) \in M(n \times k, \kappa(x))$$

is injective. Assume conversely that $\mathcal{J}_x$ is injective, then the next lemma implies that the map $B\{f_1, ..., f_k\} \to I/I^2 \xrightarrow{d} B\{dx_1, ..., dx_n\}$ is also split injective in a neighborhood of x. If the niegborhood is small enough so that the first map is surjective, this then proves that also $I/I^2 \xrightarrow{d} B\{dx_1, ..., dx_n\}$ is split injective in this neighborhood (this follows from a little diagram chase).

Lemma 6.2. *Let A be a ring and $M : A^k \to A^n$ be an A-linear map. If $M \otimes_A \kappa(x)$ is injective for some $x \in \mathrm{Spec}(A)$, then M is split injective in a neigborhood of x, i.e there exists an $f \notin x$ such that $M[f^{-1}]$ is split injective.*

Proof. After reordering we may assume that

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

with M_1 a $k \times k$-matrix which is invertible after base-change to $\kappa(x)$. Equivalently the determinant $f := \det(M_1) \in A$ does not map to zero in $\kappa(x)$, i.e. is not contained in x. But then M_1 is also invertible over $A[f^{-1}]$ and we get a retract of $M[f^{-1}]$ by the matrix

$$\begin{pmatrix} M_1^{-1} & 0 \end{pmatrix}$$

□

[4]Note that $\kappa(x)$ is the same for Z and X and thus umambiguous

Altogether we have proven:

THEOREM 6.3. *A closed subscheme $Z \subseteq \mathbb{A}^n_k$ of is smooth precisely if it is locally of finite presentation and if around every point $x \in Z$ there exist $f_1, ..., f_k \in k[x_1, ..., x_n]$ with $Z = V(f_1, ..., f_k)$ around x and the Jacobi matrix $\mathcal{J}_x \in M(n \times k, \kappa(x))$ (which depends on the $f_i's$) has rank k. Equivalently the vectors*

$$\begin{pmatrix} \partial f_i/\partial x_1 \\ \partial f_i/\partial x_2 \\ \dots \\ \partial f_i/\partial x_n \end{pmatrix}$$

for $i = 1, ..., k$ are linearly independent in $\kappa(x)^n$.

Note that any scheme which is locally of finite presentation is locally a closed subscheme of $\mathbb{A}^n$.

REMARK 6.4. Also note that it is the number of equations in the previous theorem that matters. For example one can always add a redundand version of one f_i and then the injectivity fails. In fact, one can define the tangential codimension of Z in X at the point x as the number

$$d_x = n - \dim T_x Z.$$

which is the dimension of the kernel of the map $i_x^*\Omega^1_{\mathbb{A}^n_k/k} \to i_x^*\Omega^1_{Z/k}$ where $i_x : \mathrm{Spec}(\kappa(x)) \to X$. We know there is a surjection from $I/I^2 \otimes_k \kappa(x)$ onto this kernel and so the injectitity is equivalent to the question whether around x the subscheme Z can be described by d_x-equations. One a priori knows by the previous discussion that one needs at least d but it's not clear that one needs exactly d. We will soon translate this observation in a statement about Krull-dimensions.

THEOREM 6.5. *Let $f : X \to S$ be a map of locally finite presentation. Then the following are equivalent:*

(1) *f is smooth.*
(2) *f is flat and has smooth fibres, that is for any point $s \in S$ with corresponding map $\mathrm{Spec}(\kappa(s)) \to S$ we have that $X \times_S \mathrm{Spec}(\kappa(s)) \to \mathrm{Spec}(k)$ is smooth.*
(3) *f is flat and has smooth geometric fibres, that is for map $\mathrm{Spec}(k) \to S$ with k an algebraically closed field we have that $X \times_S \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is smooth.*

PROOF. Proof omitted. The idea for (1) $\Rightarrow$ (2) is to use the standard description of smooth schemes by the Jacobi criterion (so called standard smooth) and show flatness for such concrete quotients of polynomial rings by hand (which is still quite some work). The fact that the fibres are smooth is clear since it is the pullback of smooth maps.
For (2) $\Rightarrow$ (1) one uses the tangential criterion Proposition 6.1 for smootheness again (after locally embedding in some $\mathbb{A}^n_k$). Then by flateness this can be reduced to residue fields using Lemma 6.2 and thus we get the statement we need.
(2) $\Rightarrow$ (3) is clear and for (3) $\Rightarrow$ (2) one uses descent. $\square$

7. Smooth maps are open

We now want to draw some conclusions from that, specifically that smooth maps are open. This requires some work.

DEFINITION 7.1. *We say that a map $R \to S$ of rings satisfies* going down *if for an inclusion of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}'$ in R and a prime ideal $\mathfrak{q}' \subseteq S$ lying over $\mathfrak{p}'$ there exists a prime ideal $\mathfrak{q} \subseteq \mathfrak{q}'$ lying over $\mathfrak{p}$.*

Geomerically this means that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ has the property that if we have two points $x, x' \in \mathrm{Spec}(R)$ with $x \rightsquigarrow x'$ (i.e. x' is a specialization of x meaning that x' lies in the closure of x) and a preimage y' of x' we find a preimage y of x such that $y \rightsquigarrow y'$.
Yet another reformulation is that for any pair of prime ideals $\mathfrak{p}' \subseteq R$ and $\mathfrak{q}' \subseteq S$ lying over $\mathfrak{p}'$ we have that the induced map

$$\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R_{\mathfrak{p}'})$$

is surjective.

EXAMPLE 7.2. Let $R \to S$ be flat. Then it satisfies going down. To see this note that the induced map

$$\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R_{\mathfrak{p}'})$$

is flat as well. But flat maps between local rings are automatically faithfully flat (exercise), hence surjective on prime spectra.

LEMMA 7.3. *Assume the map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open. Then $R \to S$ satisfies going down.*

PROOF. Assume that $\mathfrak{p} \subseteq \mathfrak{p}'$ in R and $\mathfrak{q}' \subseteq S$ lies over $\mathfrak{p}'$. For every $s \in S$ with $s \notin \mathfrak{q}'$ we see that $\mathfrak{p}$ is in the image of $D(s) \subseteq \mathrm{Spec}(S)$ by openness of f. Thus

$$S[s^{-1}] \otimes_R \kappa(\mathfrak{p}) \neq 0 \ ,$$

The stalk $S_{\mathfrak{q}'}$ is the filtered colimit of the $S[s^{-1}]$ and thus we see that that

$$S_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p}) \neq 0 \ .$$

Thus $\mathfrak{p}$ is the image of the $\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R)$ as desired. □

The next goal is to prove a converse, namely that maps which satisfy going down and are of finite presentation are open.

LEMMA 7.4. *Compositions of maps which satisfy going down also satisfy going down.*

PROOF. Clear by definition. □

DEFINITION 7.5. *A subset $E \subseteq X$ of a topological space X is called retrocompact if the inclusion map $E \to X$ is quasi-compact. In other words: for every quasi-compact open $U \subseteq X$ the intersection $U \cap E$ is quasi-compact.*

EXAMPLE 7.6. Finite unions of retrocompact subsets are retrocompact. This follows by the usual rule

$$U \cap (V_1 \cup ... \cup V_n) = (U \cap V_1) \cup (U \cap V_2) \cup ... \cup (U \cap V_n)$$

and the fact that finite unions quasi-compacts are quasi-compact.
Similarly finite intersections of retrocompact opens are also retrocompact. To see this let U be quasi-compact and V_i be retrocompact. Then note that

$$U \cap (V_1 \cap ... \cap V_n) = ((U \cap V_1) \cap V_2) \cap .... \cap V_n$$

and the first intersection is quasi-compact since V_1 is retrocompact. Then the second intersection is quasi-compact since V_2 is retrocompact etc.

EXAMPLE 7.7. Assume that $X = \mathrm{Spec}(R)$. Then we claim that an open subset $V \subseteq X$ is retrocompact iff it is quasi-compact. To see this first note that if V is retrocompact then since $\mathrm{Spec}(R)$ is quasi-compact we see that $V = V \cap \mathrm{Spec}(R)$ is quasi-compact. Conversely every quasi-compact open is the finite union of $D(f) = \mathrm{Spec}(R[1/f])$ and so the claim reduces to showing that $D(f) \to X$ is quasi-compact, which is true since both are affine (see Example 3.8).

DEFINITION 7.8. *A subset $E \subseteq X$ of a topological space is called constructible if it can be written as a finite union of subsets of the form*

$$U \cap V^c$$

for $U, V \subseteq X$ retrocompact opens.

PROPOSITION 7.9. *The set of constructible subsets of X is the smallest class of subsets that contains the retrocompact opens and is closed under finite unions, intersections and complements (i.e. the Boolean algebra generated by the retrocompact opens).*

PROOF. Closure of constructible sets under finite unions is clear. The emptyset and X are clearly constructible since they are retrocompact. Let

$$E = \bigcup_{i=1}^{n} U_i \cap V_i^c \qquad E' = \bigcup_{i=1}^{n} U_i' \cap (V_i')^c$$

be constructible, then the intersection is

$$E \cap E' = \bigcup_{i,.j} U_i \cap V_i^c \cap U_i' \cap (V_i')^c = \bigcup_{i,.j} (U_i \cap U_i') \cap (V_i \cup V_i')^c$$

and $U_i \cap U_i'$ as well as $V_i \cap V_i^c$ are retrocompact. The case of complements works similar. The fact that the construcibles are the minimal such set is clear. ☐

THEOREM 7.10. *(Chevalley) Suppose that $R \to S$ is of finite presentation. Then the image of a constructible subset of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is constructible, in particular the image is constructible. More generally a map $f : X \to Y$ of schemes preserves constructible subsets if it is of finite presentation (i.e. quasi-compact and locally of finite presentation).*

EXAMPLE 7.11. In particular the image of a scheme might not be a scheme. For example the morphism

$$f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$$

informally given by $(x, y) \mapsto (x, xy)$ and precisely given by the ring map $\mathbb{C}[s, t] \to \mathbb{C}[s, t]$ with $s \mapsto s$ and $t \mapsto st$. This has image given by the union

$$\{x \neq 0\} \cup \{x = y = 0\}$$

or more precisely the union of the open set $D(s)$ and the point at the origin. To see this we note that the map f takes $D(s)$ homeomorphically to $D(s)$ (and the preimage of $D(s)$ is $D(s)$) and is constant on the complement $V(s) \subseteq \mathbb{A}^2$. This is not a subscheme but constructible.

Our goal is to prove Chevalley's theorem. For this we will need some auxiliary statements. For the next few lemmas we will R be an arbitrary ring.

LEMMA 7.12. *Let $I \subseteq R$ be a finitely generated ideal. Then the map $i : \mathrm{Spec}(R/I) \to$* $\mathrm{Spec}(R)$ *sends constructible sets to constructible sets.*

PROOF. If $I = (f_1, ..., f_n)$ then the image $V(I)$ is the complement of the union of the $D(f_i)$'s, hence constructible. We now need to show that i sends $U \cap V^c$ for $U, V \subseteq \mathrm{Spec}(R/I)$ retrocompact open to a constructible subset of $\mathrm{Spec}(R)$. Since i is injective it also preserves intersections, thus we can reduce to showing it for U and V^c and since the image is constructible even to the case of a retrocompact open U (the image of V^c is the intersection of $i(V)^c \cap \mathrm{Im}(i)$.
But retrocompact opens are quasi-compact, hence finite unions of principal opens $D(r_i)$ with $r_i \in R/I$. Thus we can reduce to the case $D(r_i)$. For any lift r_i' of r_i we find that

$$i(D(r_i)) = D(r_i') \cap \mathrm{im}(i)$$

which is constructible. □

LEMMA 7.13. *Inclusions of principal opens* $\mathrm{Spec}(R[x^{-1}]) \to \mathrm{Spec}(R)$ *send constructible sets to constructible sets.*

PROOF. We argue as before that the intersection in the small set agrees with the intersection in the bigger one. Moreover the image is constructible. Therefore it suffices to show that retrocompact opens are send to constructible ones, which is clear since these are precisely the quasi-compact ones. □

LEMMA 7.14. *For the inclusion* $R \to R[x]$ *the induced map* $f : \mathbb{A}^1_R \to \mathrm{Spec}(R)$ *is open and the image of quasi-compact opens is quasi-compact open.*

PROOF. It suffices to show that the image of $D(p) \subseteq \mathbb{A}^1_R$ is quasi-compact open for some polynomial

$$p = a_0 x^d + ... + a_d\ .$$

We claim that this image is given by $D(a_0) \cup D(a_1) \cup ... \cup D(a_d)$, i.e. those prime ideals in R which do not contain any of the coefficients of f.
To see this we first claim that

$$\mathrm{Im}(f) \subseteq D(a_0) \cup D(a_1) \cup ... \cup D(a_d)$$

That is if $\mathfrak{p} \subseteq R[x]$ is a prime ideal that does not contain p then the intersection $\mathfrak{p} \cap R$ is an ideal which does not contain all the a_k's. This follows since if all the a_k's were in this intersection then $\mathfrak{p}$ would contain all the a_k's and thus also p in contradiction to the assumption.
To see that $D(a_0) \cup D(a_1) \cup ... \cup D(a_d) \subseteq \mathrm{Im}(f)$ we assume that we have an ideal $\mathfrak{p} \subseteq R$ that does not contain one of the a_k's. Then we form the ideal

$$\mathfrak{p}' = \mathfrak{p}[x] \subseteq R[x]$$

of all polynomials with coefficients in $\mathfrak{p}$. This is a prime ideal since the quotient is $R/\mathfrak{p}[x]$ is integral. Moreover we have that $\mathfrak{p}' \cap R = \mathfrak{p}$. Finally we note that $\mathfrak{p}'$ does not contain p since $\mathfrak{p}$ does not contain all the coefficients. □

LEMMA 7.15. *For the inclusion* $R \to R[x]$ *the induced map* $f : \mathbb{A}^1_R \to \mathrm{Spec}(R)$ *sends constructible sets to constructible sets.*

PROOF. Decomposing by finite unions we can immediately reduce to showing that the constructible set

$$T = D(p) \cap V(g_1, ..., g_n)$$

for $p, g_1, ..., g_n \in R[x]$ is send to a constructible set in $\mathrm{Spec}(R)$.

Note that for some $c \in R$ we can decompose $\mathrm{Spec}(R) = V(c) \sqcup D(c) = \mathrm{Spec}(R/c) \sqcup \mathrm{Spec}(R[c^{-1}])$ and similarly

$$\mathrm{Spec}(R[x]) = \mathrm{Spec}(R/c[x]) \sqcup \mathrm{Spec}(R[x^{-1}][x]) \ .$$

The image of T in both sets has the same shape with p, g_i replace by their image in $R/c[x]$ and $R[x^{-1}][x]$. Moreover to check that a subset of $\mathrm{Spec}(R)$ is constructible we can check if its intersection with $\mathrm{Spec}(R/c[x])$ and $\mathrm{Spec}(R[x^{-1}][x])$ is constructible (by Lemmas 7.12 and 7.13).
We assume that

$$\deg(g_1) \leq ... \leq \deg(g_n)$$

and want to show that $f(T)$ is constructible. We proceed by induction over the number n and the degree's of the g_i's. Let

$$g_1 = cx^d + a_{d-1}x^{d-1} + ... \ .$$

Then using the above observation we can reduce to the case where either c is zero and thus g_1 has lower degree or c is invertible. If c is invertible and $n > 1$ then we can replace g_2 by

$$g_2' = g_2 - \lambda/cx^{\deg(g_2)-\deg(g_1)}g_1$$

with $V(g_1, ..., g_n) = V(g_1, g_2', ..., g_n)$ and g_2' has strictly lower degree. The base case of the induction is the case where

$$T = D(p) \cup V(g)$$

where g has invertible leading coefficient or $T = D(p)$ the second case is covered by Lemma 7.14 and the first one by the next lemma. □

Lemma 7.16. *Let $T = D(p) \cup V(g) \subseteq \mathrm{Spec}(R[x])$ for polynomials p and g where g has an invertible leading coefficient. Then the map $f : \mathrm{Spec}(R[x]) \to \mathrm{Spec}(R)$ sends T to a set of the form $\cup_{i=1}^n D(r_i)$.*

Proof. This is Lemma 10.29.9. in the stacks project, see there for more details. Consider the R-module $M := R[x]/g$ which is finite free with basis $1, x, ..., x^{n-1}$ with $n = \deg(g)$ since the leading coefficient is a unit. Now consider the R-linear map $F : M \to M$ given by multiplication with f and let

$$\chi(t) = \det(\mathrm{id}t - F) = t^n + r_1t^{n-1} + ... + r_n$$

be its characteristic polynomial. Then the coefficients r_i do the job which follows from the following observations:

(1) Let $R \to S$ be a map of rings. Then $\mathfrak{p} \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ iff the ring

$$S \otimes_R \kappa(\mathfrak{p})$$

is non-trivial. To see this simply consider the pullback corresponding to this tensor product.

(2) The constructible set T is the spectrum of the ring

$$(R[x]/g)[f^{-1}]$$

so that the question whether $\mathfrak{p}$ lies in the image is equivalent to the question if

$$(R[x]/g)[f^{-1}] \otimes_R \kappa(\mathfrak{p}) \neq 0 \ .$$

(3) We have that
$$(R[x]/g)[f^{-1}] \otimes_R \kappa(\mathfrak{p}) = (\kappa(\mathfrak{p})[x]/g)[f^{-1}]$$
so the question becomes whether or not f acts nilpotently on $M \otimes_R \kappa(p)$.
(4) An endomorphism of a finite dimensional vector space is nilpotent iff its characteristic polynomial is of the form t^n.
(5) We have that the characteristic polynomial of $F \otimes_R \kappa(p)$ is the image of the $\chi_F(t)$ under the map
$$R[t] \to \kappa(p)[t]$$
This is naturality of the determinant.

Taking everything together we see that $\mathfrak{p}$ is not in the image iff the image of $\chi_A(t)$ in $\kappa(p)[t]$ is t^n, which is equivalent to the assertion that all coeffiicents are zero, i.e. $p \in V(r_1, ..., r_n)$.

$\square$

Proof of Theorem 7.10. We consider $S = R[x_1, ..., x_n]/(f_1, ..., f_k)$. Then we can write the map as the composition
$$R \to R[x_1] \to R[x_1, x_2] \to .... \to R[x_1, ..., x_n] \to R[x_1, ..., x_n]/(f_1, ..., f_k)$$
so that it suffices to prove the result for these maps separately. These cases are covered by the previous lemmas. $\square$

Lemma 7.17. *Let $E \subseteq \mathrm{Spec}(R)$ be constructible.*
(1) *Every element $x \in \overline{E}$ is the specialisation of some element of E.*
(2) *E is closed iff E is stable under specialization.*
(3) *E is open iff E is stable under generalization.*

Proof. Assume that $x \in \overline{E}$. Since closures of finite unions are the union of the closures we may without loss of generality assume that $E = U \cap V^c$ for $U, V \subseteq \mathrm{Spec}(R)$ retrocompact open, which is by Example 7.7 equivalently to quasi-compact open.
Writing U as a finite union of principal opens we can further assume that $U = D(r)$ is principal open and $V^c = V(I)$ for some ideal I. Now we let $U_j = \mathrm{Spec}(R[f_j^{-1}])$ be the system of principal open neighborhoods of x. Then
$$U_j \cap E \neq \emptyset$$
by assumption since x is the in the closure. Then finite intersections of U_j are again of the form U_j and thus we have that all finite intersections of $U_j \cap E$ are non-empty. Now we note that
$$\bigcap_j U_i \cap E = \mathrm{Spec}(R/I[r^{-1}, f_j^{-1} \mid j \in J])$$
and we claim that this ring $R/I[r^{-1}, f_j^{-1} \mid j \in J]$ is non-zero. This follows since it is a filtered colimit of localizations at finitely many f_j's which are non-zero and thus it is non-zero (otherwise the element 1 would have to vanish in a finite stage). This however shows that there is an element $y \in \bigcap_j U_i \cap E$. This means that x is in the closure of y, i.e. a specialization of y.
Assertion (2) is immediate: clearly closed maps are closed under specialization and the converse follows from part (1). Assertion (3) follows from (2) by passing to complements. $\square$

PROPOSITION 7.18. *Let $f : R \to S$ be a map which satifies going down and is of finite presentation. Then the map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is open.*

PROOF. We need to show that the image of open sets is open. We can immediately reduce this to principal opens $\mathrm{Spec}(S[s^{-1}]) \subseteq \mathrm{Spec}(S)$. But then we can replace S by $S[s^{-1}]$ since the composite $R \to S \to S[s^{-1}]$ is still finitely presented and satisfies going down by Example 7.2 and 7.4. Therefore we reduce the claim without loss of generality to showing that the image of f is open.
If f satisfies going down then the image of the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed under generalization. By Chevalley's theorem it is also constructible. Thus by Lemma 7.17 it is open. □

COROLLARY 7.19. *Let $f : X \to Y$ be a flat map between schemes which is locally of finite presentation. Then f is open.*

PROOF. Being open is a local property, so that by the definition of flatness and Example 7.2 this reduces to the previous claim. □

We warn the reader that under the assumptions of the previous corollary it is not necessarily true that the image of constructible sets are constructible.

COROLLARY 7.20. *Smooth and étale maps of schemes are open.*

PROOF. Smooth and étale maps are flat and locally of finite presentation. Thus they are open by the previous claim. □

8. Smooth schemes over fields

Now our goal is to provide criteria for schemes to be smooth. The key will be the notion of regularity from algebra which we introduce now.

THEOREM 8.1 (Krull). *Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Then*

$$\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$$

where $\dim R$ *is the Krull dimension.*

REMARK 8.2. By Nakayma the dimension of $\mathfrak{m}/\mathfrak{m}^2$ equals the mininal numbers of generators of $\mathfrak{m}$. In particular the dimension of local Noetherian rings is finite since . We warn the reader the that dimensional of general noetherian rings might be infinite.

DEFINITION 8.3. *A regular local ring is a local, noetherian ring R such that*

$$\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2 \ .$$

PROPOSITION 8.4. *For a local noetherian ring, the following are equivalent:*

(1) *R is regular*
(2) *$\mathfrak{m}$ can be generated by* $\dim R$ *elements.*
(3) *The graded k-algebra*

$$\bigoplus_{k=0}^{\infty} \mathfrak{m}^k/\mathfrak{m}^{k+1}$$

is equivalent to $k[x_1, ..., x_n]$ with x_i in degree 1.

EXAMPLE 8.5. Let R be a noetherian local ring of $\dim(R) = 0$. Then R is local iff R is a field. An example of a non-regular local ring of dimension 0 is for example $k[x]/x^2$ for a field k or $\mathbb{Z}/4$.
If $\dim(R) = 1$ then R is regular iff the maximal ideal is a principal ideal. For example this is the case for $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$.

LEMMA 8.6. *Regular local rings are reduced and* $\bigcap \mathfrak{m}^n = 0$.

THEOREM 8.7. *Let* $X \to \mathrm{Spec}(k)$ *be a scheme over an algebraically closed field* k. *Then the following are equivalent:*

(1) $X \to \mathrm{Spec}(k)$ *is smooth*
(2) $X \to \mathrm{Spec}(k)$ *is regular, that is each local ring* $\mathcal{O}_{X,x}$ *is a regular local ring.*
(3) $X \to \mathrm{Spec}(k)$ *is locally of finite type,* $\Omega^1_{X/k}$ *is a vector bundle and* X *is reduced.*

9. Sheaf cohomology

The goal of sheaf cohomology is to construct abelian groups

$$H^i(X, \mathcal{A}) \qquad \text{for } i \in \mathbb{N}$$

with $H^0(X, \mathcal{A}) = \Gamma(X, \mathcal{A})$ and such that for every exact sequence $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ we get a long exact sequence

$$0 \to H^0(X, \mathcal{A}_1) \to H^0(X, \mathcal{A}_2) \to H^0(X, \mathcal{A}_3) \to H^1(X, \mathcal{A}_1) \to H^1(X, \mathcal{A}_2) \to H^1(X, \mathcal{A}_3) \to H^2(X, \mathcal{A}_2) \to$$

DEFINITION 9.1. *A δ-functor (from sheaves on X to abelian groups) is given by a sequence of product preserving functors*

$$F^i : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab} \qquad i \in \mathrm{N}$$

together with a natural maps

$$\delta^i : F^i(\mathcal{A}_3) \to F^{i+1}(\mathcal{A}_1)$$

for every short exact sequence $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ such that the induced sequence

$$0 \to F^0(\mathcal{A}_1) \to F^0(\mathcal{A}_2) \to F^0(\mathcal{A}_3) \to F^1(\mathcal{A}_1) \to ..$$

is long exact. A morphism of delta functors $(F^i, \delta^i) \to (G^i, \delta^i)$ is given by a sequence of natural transformations $F^i \to G^i$ such that the diagram

$$\begin{tikzcd} F^i \arrow[r] \arrow[d, "\delta"] & G^i \arrow[d, "\delta"] \\ F^{i+1} \arrow[r] & G^{i+1} \end{tikzcd}$$

commutes. There is an obvious notion of morphisms of δ-functors.

The main result about sheaf cohomology that we want to eventually prove is the following:

THEOREM 9.2 (Grothendieck). *For every space X there exists an initial δ-functor (H^i, δ^i) with $H^0(\mathcal{A}) \cong \mathcal{A}(X)$. We write*

$$H^i(X, \mathcal{A})$$

and call it the i-th sheaf cohomology of X with value in $\mathcal{A}$.
Moreover in degree 1 it agrees with our H^1 and δ^0 agrees with our δ. For affine schemes X and quasi-coherent sheaves $\mathcal{A}$ we have that $H^i(X, \mathcal{A}) = 0$ for $i > 0$.

Note that being initial of course uniquely determines this functor.

Definition 9.3. *An object $A \in \mathcal{C}$ in a category is called* injective *if every monomorphism $i : I \to J$ and every morphism $I \to A$ there exists an extension $J \to A$, i.e.*

$$\begin{tikzcd} I \arrow[r] \arrow[d, tail] & A \\ J \arrow[ur, dashed] & \end{tikzcd}$$

Example 9.4. (1) a set is injective in the category of sets iff it is non-empty.

(2) Every vector space over a field k is injective. To see this assume hat $V \to V'$ is an injection. Then we choose a complement to V in V' and take the retraction.
(3) The abelian group $\mathbb{Z}$ is not injective, since $\mathbb{Z} \to \mathbb{Q}$ does not admit a section. The groups $\mathbb{Z}/n$ is also not injective since the inclusion $\mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ sending 1 to $1/n$ does not admit a section.
(4) In any category, products of injective objects are injective.

Remark 9.5. (1) The definition of injective means that $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ sends monomorphisms in $\mathcal{C}$ to epimorphisms of sets.

(2) Assume that the ambient category $\mathcal{C}$ has the property that monomorphisms are stable under pushout. This is for example the case for the category of sets or for any abelian category but not in all categories. Then an object A is injective iff every monomorphism $i : A \to A'$ has a retract, i.e. a morphism $r : A' \to A$ such that $r \circ i = \mathrm{id}_A$. We will freely switch between these equivalent descriptions.

   To see that this is indeed equivalent assume first that A is injective. Then we get a retract by lifting against

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, tail] & A \\ A' \arrow[ur, dashed] & \end{tikzcd}$$

   Conversely assume that monomorphisms out of A admit retracts and we have a lifting diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d, tail] & A \\ J & \end{tikzcd}$$

   Then for the pushout $A' = A \coprod_I J$. The morphism $A \to A'$ is a monomorphism by assumption on the category $\mathcal{C}$, thus admits a retract r. But by the universal property of the pushout the retract in particular yields a lift.
(3) Sometimes one modifies the definition if injective by requiring lifts only for specific classes of monomorphisms (or generally classes of morphisms), e.g. so-called regular monomorphisms. In our cases of interest this will not make a difference so that we do not have to go into these subtleties here.

Theorem 9.6. *For every abelian group A there is an injection $A \to A'$ with A' an injective abelian group. Moreover an abelian group A is injective, precisely if it is divisible, that is if for every $a \in A$ and $n \in \mathbb{Z}$ there exists an object b with $nb = a$*

Proof. Assume that A is injective and fix $a \in A$ and consider the injection $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. Then the induced map

$$A = \mathrm{Hom}(\mathbb{Z}, A) \to \mathrm{Hom}(\mathbb{Z}, A) = A$$

which is given by multiplication with n. It is surjective by injectivity of A. Thus A is divisible.
Assume conversely that A is divisible and choose an injection $A \to A'$. Consider the set of pairs consisting of an abelian group $A \subseteq B \subseteq A'$ and a retract br of the inclusion $A \to B$. Such pairs (B, r) are ordered by

$$(B, r) \leq (B', r') \leftrightarrow B \subseteq B' \qquad \text{and} \qquad r'|_B = r\ .$$

If we have a totally order collection of pairs (B_i, r_i) then we get a new pair $(\bigcup B_i, r)$ where r is defined as the unique extension of the r_i's. Thus Zorn's Lemma applies and we get a maximal element (B, r). We want to show that $B = A'$. Assume not and pick $x \in A' \setminus B$. If for all $n \geq 1$ we have $nx \notin B$ then we have that the subgroup $\langle B, x\rangle \subseteq A'$ is isomorphic to $B \oplus \langle x\rangle$ and thus we could extend r (e.g. by setting $r(x) = 0$ in contradiction to maximality. Thus there exists $n \geq 1$ with $nx \in B$ and we assume that n is minimal, so that

$$\langle B, x\rangle \cong B \oplus \langle x\rangle/(nx, -nx)\ .$$

Now consider the element $r(nx)$. By divisibility of A there is an element a with

$$na = r(nx)\ .$$

We now define a morphism

$$r' = \langle B, x\rangle \to A$$

as r on B and $r'(x) = a$. This is well-defined in contradiction to maximality of B which shows that A was injective.
Now to embedd an arbitrary group A into an injective abelian group let $0 \neq x \in A$ be an element of order n (where $n \in [1, \infty]$). We then consider the abelian group

$$A_x := \begin{cases} \mathbb{Q}/n\mathbb{Z} & \text{for } n < \infty \\ \mathbb{Q} & \text{for } n = \infty \end{cases}$$

There is a map $\langle x\rangle \to \mathbb{A}_x$ sending x to 1. Since A_x is injective we extend this to a map $A \to A_x$ that still sends x to 1. Now consider

$$A \to \prod_{x \neq 0} A_x$$

which is injective (since every element is non-zero in at least one factor) with divisible target (since products of divisible groups are clearly also divisible). $\square$

Note that in the previous proof we have shown that each abelian group injects into a product of $\mathbb{Q}$'s and $\mathbb{Q}/\mathbb{Z}$'s (as $\mathbb{Q}/n\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$). However we can also embedd $\mathbb{Q}$ into a product of $\mathbb{Q}/Z$'s by the map

$$\mathbb{Q} \to \prod_n \mathbb{Q}/n\mathbb{Z} \qquad q \mapsto ([q])_{n \in \mathrm{N}}$$

which is injective. It follows that every abelian group injects into a product of $\mathbb{Q}/\mathbb{Z}$'s. Moreover if A is injective then every such inclusion has a retract, thus the injective abelian groups are precisely those which are retracts of products of $\mathbb{Q}/\mathbb{Z}$'s.

COROLLARY 9.7. *For any module M over a ring R there is an injection $M \to M'$ into an injective R-module M'.*

PROOF. Consider the underlying abelian group of M and pick an injection $j : M \to A$ to an injective abelian group. We now set

$$M' := \mathrm{Hom}_{\mathrm{Ab}}(R, A)$$

the R-module of maps of abelian groups $R \to A$. This is an R-module by acting on the source, i.e.

$$(r \cdot \phi)(x) = \phi(rx).$$

Now there is an R-linear map

$$i : M \to M' \qquad m \mapsto \phi_m \qquad \phi_m(r) = j(rm) \ .$$

such that j is the composite $M \to M' \xrightarrow{ev_1} A$. Therefore i is injective. Moreover we claim that M' is injective as an R-module. To see this note that

$$\mathrm{Hom}_R(I, \mathrm{Hom}_{\mathrm{Ab}}(R, M)) \cong \mathrm{Hom}_{\mathrm{Ab}}(I, M) \qquad \mathrm{Hom}_R(J, \mathrm{Hom}_{\mathrm{Ab}}(R, M)) \cong \mathrm{Hom}_{\mathrm{Ab}}(J, M)$$

and for every monomorphism $I \to J$ the induced map under this equivalence is precomposition. So injectivity follows from injectivity of M as an abelian group. □

Again the proof really shows (with the comments before the corollary) that every R-module embedds into a product of $\mathrm{Hom}_{\mathrm{Ab}}(R, \mathbb{Q}/Z)$. Moreover every injective module is a retract of those.

THEOREM 9.8 (Enough injectives). *Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X or more generally a sheaf of modules over some sheaf of rings. Then there exists a monomorphism $\mathcal{F} \to \mathcal{F}'$ where $\mathcal{F}'$ is injective.*

PROOF. For every point $x \in X$ we denote the inclusion from the one point space by

$$i_x : \mathrm{pt} \to X$$

A sheaf on pt is simply given by an abelian group. Now assume that we pick an injective abelian group A. Then we claim that

$$(i_x)_*(A) \in \mathrm{Shv}(X, \mathrm{Ab})$$

is injective. To see this we consider a monomorphism $F \to G$ of sheaves and want to prove that

$$\mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab})}(G, (i_x)_*(A)) \to \mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab})}(F, (i_x)_*(A))$$

is surjective. Using the adjunction $i_x^{-1} \dashv (i_x)_*$ this translates into the map

$$\mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab})}(G_x, A) \to \mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab})}(F_x, A)$$

being surjective. But the map $F_x \to G_x$ is also injective so that this follows from injectivity of A.
Now let $\mathcal{F}$ be an arbitrary sheaf of abelian groups. For every point $x \in X$ we pick an injection $\mathcal{F}_x \to A_x$ and set

$$\mathcal{F}' := \prod (i_x)_*(A_x)$$

and define a map $\mathcal{F} \to \mathcal{F}'$ as the collection of maps $\mathcal{F} \to (i_x)_*(A_x)$ adjoint to the maps $\mathcal{F}_x \to A_x$. This map is a monomorphism as we can check this on stalks where it follows by definition since the stalk $\mathcal{F}'_x$ maps to A_x. □

Again we see that we can embedd every sheaf in fact into a product of $(i_x)_*\mathbb{Q}/Z$'s for varying points x.
We are still in the process of proving Theorem 9.2.

Definition 9.9. *Let $G : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$ be an additive functor, i.e. a functor which preserves products. The G is called* effaceable *if for every sheaf F there exists a monomorphism $\mathcal{F} \to \mathcal{F}'$ such that $G(\mathcal{F}') = 0$.*

Lemma 9.10. *An additive functor G is effaceable iff for each injective sheaf $\mathcal{I}$ we have $G(\mathcal{I}) = 0$.*

Proof. One direction immediately follows from Theorem 9.8. Conversely if G is effaceable and $\mathcal{I}$ is an injective sheaf. Then we can choose some monomorphism $\mathcal{I} \to \mathcal{I}'$ with $G(\mathcal{I}') = 0$. But since $\mathcal{I}$ is injective we get a retract $\mathcal{I}' \to \mathcal{I}$ exhibiting $\mathcal{I}$ as a retract of $\mathcal{I}'$. Thus $G(\mathcal{I})$ is a retract of $G(\mathcal{I}') = 0$ and therefore $G(\mathcal{I}) = 0$. $\square$

Definition 9.11. *A δ-functor (R^i, δ^i) is called* effaceable *if all R^i for $i > 0$ are effaceable.*

Proposition 9.12. *Every effaceable δ-functor (R^i, δ^i) is initial among all δ-functors with $G^0 \cong R^0$.*

Proof. Assume that we are given a δ-functor (G^i, δ^i) with an isomorphism $G^0 \cong R^0$. We need to show that there is a unique map of δ-functors

$$\varphi^i : (R^i, \delta^i) \to (G^i, \delta^i) \ .$$

We will do this by induction over i, i.e. assume that the natural transformations $\varphi^k : R^k \to G^k$ have already been constructed for $k < i$ in a way that is compatible with the respective δ's.
Thus pick some sheaf $\mathcal{A}$, we want to define map

$$\varphi^i : R^i(\mathcal{A}) \to G^i(\mathcal{A}) \ .$$

To this end we pick some monomorphism $\mathcal{A} \to \mathcal{A}'$ with $R^i(\mathcal{A}') = 0$ and consider the relevant chunk of the long exact sequences

$$\begin{tikzcd}
R^{i-1}(\mathcal{A}') \arrow[r] \arrow[d, "\varphi^{i-1}"] & R^{i-1}(\mathcal{A}'/\mathcal{A}) \arrow[r, "\delta"] \arrow[d, "\varphi^{i-1}"] & R^i(\mathcal{A}) \arrow[r] \arrow[d, dashed] & R^i(\mathcal{A}') = 0 \\
G^{i-1}(\mathcal{A}') \arrow[r] & G^{i-1}(\mathcal{A}'/\mathcal{A}) \arrow[r, "\delta"] & G^i(\mathcal{A}) &
\end{tikzcd}$$

The upper map δ exhibits $R^i(\mathcal{A})$ as the cokernel of the map $R^{i-1}(\mathcal{A}') \to R^{i-1}(\mathcal{A}'/\mathcal{A})$. Thus there is a unique dashed morphism by the universal property of the cokernel together with the map the the composition from the upper left term to the lower right term is zero, since the lower composition is zero.
We were forced to define φ^i this way, but it is not clear that the φ^i defined this way is well-defined (as it might depend on the choice of $\mathcal{A} \to \mathcal{A}'$). If we have a map diagram $\mathcal{A} \to \mathcal{A}' \to \mathcal{A}''$ with $R(\mathcal{A}'') = 0$ as well, the map $R^i(\mathcal{A}) \to G^i(\mathcal{A})$ obtained from the pair $\mathcal{A} \to \mathcal{A}'$ agrees with the one obtained from the pair $\mathcal{A} \to \mathcal{A}''$, by looking at the resulting commutative diagram. Finally, for two different choices

$\mathcal{A} \to \mathcal{A}'$ and $\mathcal{A} \to \mathcal{A}''$ we find that the pushout

$$\begin{array}{ccc} \mathcal{A} & \rightarrowtail & \mathcal{A}' \\ \downarrow & & \downarrow \\ \mathcal{A}'' & \rightarrowtail & P \end{array}$$

can also be embedded into a sheaf $P \to \mathcal{A}'''$ on which R^i vanishes. So $\mathcal{A}'$ and $\mathcal{A}'''$ lead to the same map, and $\mathcal{A}''$ and $\mathcal{A}'''$ do, so we see that $\mathcal{A}'$ and $\mathcal{A}''$ lead to the same map.
Similarly, we see that the map φ^i is really a natural transformation: For $\mathcal{A} \to \mathcal{B}$, we always find $\mathcal{A}'$ and $\mathcal{B}'$ on which R^i vanishes, which fit into a diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A}' & \longrightarrow & \mathcal{B}' \end{array}$$

which shows that φ^i is natural with respect to $\mathcal{A} \to \mathcal{B}$.
Finally one needs to show that the maps defined this way constitute a map of δ-functors, i.e. are compatible with morphisms δ. To this end we assume that we have a short exact sequence

$$0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to 0$$

of sheaves. Then we need to show that the diagram

$$\begin{array}{ccc} R^{i-1}(\mathcal{A}_3) & \xrightarrow{\delta} & R^i(\mathcal{A}_1) \\ \downarrow{\scriptstyle \varphi^{i-1}} & & \downarrow{\scriptstyle \varphi^i} \\ G^{i-1}(\mathcal{A}_3) & \xrightarrow{\delta} & G^i(\mathcal{A}_1) \end{array}$$

commutes. By construction, we do know that this commutes for exact sequences where $R^i(\mathcal{A}_2) = 0$. We may pick $\mathcal{A}_2 \to \mathcal{A}_2'$ with that property and get a new sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathcal{A}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2' & \longrightarrow & \mathcal{A}_3' & \longrightarrow & 0. \end{array}$$

Looking at the resulting cube, we are done. □

Proposition 9.13. *For any left exact functor* $F : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$ *an effaceable* δ*-functor* (RF^i, δ^i) *extending* F *exists.*

Proof idea. We will now see how $RF^i(\mathcal{F})$ are determined inductively. Well-definedness will follow later. First we choose a monomorphism

$$\mathcal{F} \to \mathcal{I}^0$$

into an injective sheaf, with cokernel $\mathcal{F}^1 := \mathcal{I}^0/\mathcal{F}$. Then since we want $RF^1(\mathcal{I}^0) = 0$ and an exact sequence

$$RF^0(\mathcal{I}^0) \to RF^0(\mathcal{F}^1) \xrightarrow{\delta} RF^1(\mathcal{F}) \to RF^1(\mathcal{I}^0) = 0$$

we need to have

$$RF^1(\mathcal{F}) := \mathrm{coker}(F(\mathcal{I}^0) \to F(\mathcal{F}^1)).$$

We also see from the rest of the long exact sequence that

$$RF^n(\mathcal{I}^0) \to RF^n(\mathcal{F}^1) \xrightarrow{\delta} RF^{n+1}(\mathcal{F}) \xrightarrow{R} F^{n+1}(\mathcal{I}^0)$$

is exact, so $RF^{n+1}(\mathcal{F}) \cong RF^n(\mathcal{F}^1)$ since the outer terms vanish. For example, to compute $RF^2(\mathcal{F}) \cong RF^1(\mathcal{F}^1)$, we pick another monomorphism $\mathcal{F}^1 \to \mathcal{I}^1$ into an injective, and set $\mathcal{F}^2$ to be the cokernel. Then we need to have

$$RF^2(\mathcal{F}) = \operatorname{coker}(F(\mathcal{I}^1) \to F(\mathcal{F}^2)) \ .$$

If we keep playing this game we obtain a diagram like this

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}^0 \arrow[rr, "d^0"] \arrow[dr, two heads] & & \mathcal{I}^1 \arrow[rr, "d^1"] \arrow[dr, two heads] & & \mathcal{I}^2 \arrow[rr, "d^2"] \arrow[dr, two heads] & & \ldots \\
& & & \mathcal{F}^1 \arrow[ur, tail] & & \mathcal{F}^2 \arrow[ur, tail] & & \mathcal{F}^3 \arrow[ur, tail] &
\end{tikzcd}$$

and we get that $RF^n(\mathcal{F}) \cong \ldots \cong RF^1(\mathcal{F}^{n-1})$ and thus

$$\begin{aligned}
RF^n(\mathcal{F}) &= \operatorname{coker}(F(\mathcal{I}^{n-1}) \to F(\mathcal{F}^n)) \\
&= \operatorname{coker}(F(\mathcal{I}^{n-1}) \to F(\ker(\mathcal{I}^n \to \mathcal{I}^{n+1})) \\
&= \operatorname{coker}(F(\mathcal{I}^{n-1}) \to \ker(F(\mathcal{I}^n) \to F(\mathcal{I}^{n+1})) \ .
\end{aligned}$$

Thus we can express all the RF^i in terms of the *cochain complex*

$$F(\mathcal{I}^0) \to F(\mathcal{I}^1) \to F(\mathcal{I}^2) \to \ldots$$

more precisely as the n-th cohomology. The main problem now is to show that the functors defined this way do not depend on the choice of the $\mathcal{I}^j$. This will be shown in the next section after establishing some abstract terminology. □

10. Abelian categories

DEFINITION 10.1. *Let $\mathcal{A}$ be a category. We say that*

(1) *$\mathcal{A}$ is pointed if there is an object $0 \in \mathcal{A}$ which is initial and terminal . In this case we have for every pair of objects $A, B \in \mathcal{C}$ a morphism $A \to 0 \to B$ which we denote by $0 \in \operatorname{Hom}_{\mathcal{A}}(A, B)$.*

(2) *$\mathcal{A}$ is semiadditive if it is pointed, has finite products and coproducts and for every pair of objects $A, B \in \mathcal{A}$ the morphism*

$$A \coprod B \to A \times B \qquad \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}$$

is an isomorphism. In this case we write $A \oplus B$ for the (co)product and call it a biproduct.

EXAMPLE 10.2. The category of pointed sets Set_* is pointed with the zero object $*$. The category of groups is pointed with zero object 1. The category of abelian monoids, abelian groups, R-modules and sheaves of abelian groups are all semiadditive with coproduct $\oplus$.

In any semiadditive category we can define an addition $+$ on $\operatorname{Hom}_{\mathcal{A}}(A, B)$ for objects $A, B \in \mathcal{A}$ by defining $f + g$ to be the morphism

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla} B \ .$$

It is straightforward to see that this gives $\mathrm{Hom}_{\mathcal{A}}(A,B)$ the structure of an abelian monoid with neutral element 0. The composition

$$\circ : \mathrm{Hom}_{\mathcal{A}}(B,C) \times \mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{A}}(A,C)$$

is bilinear. (If we write maps between direct sums as matrices, we also get the usual formula for matrix multiplication, involving this $+$)

LEMMA 10.3. *Let $\mathcal{A}$ be a semiadditive category. Then the following are equivalent:*

(1) *For every object $A \in \mathcal{A}$ the morphism*

$$A \oplus A \to A \oplus A \qquad \begin{pmatrix} \mathrm{id}_A & \mathrm{id}_A \\ 0 & \mathrm{id}_A \end{pmatrix}$$

is an isomorphism.

(2) *For every object $A \in \mathcal{A}$ there exists a morphism $i : A \to A$ such that the composition*

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\mathrm{id}_A \oplus i} A \oplus A \xrightarrow{\nabla} A$$

is zero.

(3) *For any object A the monoid $\mathrm{Hom}_{\mathcal{A}}(A,A)$ is an abelian group.*

(4) *For any pair of objects $A, B \in \mathcal{A}$ the monoid $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is an abelian group.*

PROOF. Clearly, (4) $\Rightarrow$ (3) $\Rightarrow$ (2), and for (2) $\Rightarrow$ (1) we observe that an inverse is given by

$$\begin{pmatrix} \mathrm{id}_A & i \\ 0 & \mathrm{id}_A \end{pmatrix}.$$

For (1) $\Rightarrow$ (4), we observe that on $\mathrm{Hom}(B, A\oplus A) \cong \mathrm{Hom}(B,A) \times \mathrm{Hom}(B,A)$, post-composition with the morphism from (1) acts by $(f,g) \mapsto (f+g, g)$. In particular, if the morphism from (1) is an isomorphism, $\mathrm{Hom}(B,A)$ is an abelian group, since we find an additive inverse to g by considering the preimage of $(0,g)$. ☐

DEFINITION 10.4. *If either of the equivalent conditions of the previous lemma are satisfied we call $\mathcal{A}$ additive. If $\mathcal{A}$ is additive it is called abelian if moreover for each morphism $f : A \to B$ the objects*

$$\ker(f) = \lim \left(\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \right) \qquad \textit{and} \qquad \mathrm{coker}(f) = \mathrm{colim} \left(\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd} \right)$$

exist and the canonical morphism

$$\mathrm{coker}(\ker(f) \to A) \to \ker(B \to \mathrm{coker}(f))$$

is an isomorphism. We call this object $\mathrm{im}(f)$.

EXAMPLE 10.5. The category of abelian groups is abelian, the category of abelian monoids is not additive (hence not abelian). The category of R-modules is abelian. The category $\mathrm{Shv}(X, \mathrm{Ab})$ is abelian. The category $\mathrm{QCoh}(X)$ is abelian. The category $\mathrm{Coh}(X)$ is abelian.

EXAMPLE 10.6. Consider the category of finitely generated free abelian groups. This is preadditive. Moreover we claim that it has kernels and cokernels. Kernels are clear, since subgroups of finitely generated free abelian groups are also finitely generated free. The cokernel is a bit more surprising since the the cokernel in abelian groups might have torsion, i.e. the cokernel of $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. However what we do to form the cokernel in finite free abelian groups for a morphism $f : A \to B$ is to form the cokernel in abelian groups $M := B/A$ and then take the quotient by the subgroup $M' \subseteq M$ of torsion elements. Then it is not hard to see that M/M' is indeed torsionfree and finitely generated, hence free. Moreover every morphism $M \to C$ where C is finite free sends M' to zero. This shows the universal property.
For example we have that the cokernel of $f : \mathbb{Z} \xrightarrow{n} \mathbb{Z}$ in this category is zero for each $n \neq 0$. But this category is not abelian since the canonical morphism

$$\mathbb{Z} = \operatorname{coker}(\ker(f) \to \mathbb{Z}) \to \ker(\mathbb{Z} \to \operatorname{coker}(f)) = \mathbb{Z}$$

is multiplication by n and thus not an isomorphism (note the difference to the case of abelian groups).

LEMMA 10.7. *A morphism f in an abelian category is*

1. *a monomorphism iff* $\ker(f) = 0$ *iff it is a kernel* $\ker(g) \to B$ *of some morphism $g : B \to C$.*
2. *an epimorphism iff* $\operatorname{coker}(f) = 0$ *iff it is the cokernel of some morphism $g : Z \to A$.*
3. *an isomorphism iff it is an epimorphism and a monomorphism*

Moreover monomorphisms are closed under pushouts and epimorphisms closed under pullbacks.

We note that in a general category morphisms can be epi and mono without being an isomorphism, for example the morphisms $\mathbb{Z} \to \mathbb{Q}$ is a monomorphism and epimorphism in the category of commutative rings.

PROOF. Assume that $f : A \to B$ is a monomorphism and we have any morphism $Z \to A$ such that the composition $Z \to A \to B$ is zero. Then the morphism $Z \to A$ is already zero by the definition of a monomorphism. This shows that $\ker(f) = 0$. Assume conversely that $\ker(f) = 0$. Then

$$A = \operatorname{coker}(0 \to A) = \operatorname{Im}(f) \cong \ker(B \to \operatorname{coker}(f))$$

Thus two morphisms $Z \to A$ agree precisely if the two morphisms to B agree (as morphisms into a kernel agree if they agree as morphisms to the first object by the universal property).
The proof of (2) works dually. For one direction of (3) note that isomorphisms are clearly mono and epi. For the converse assume that $f : A \to B$ is mono and epi. Then by the definition of an abelian category we get that the canonical morphism

$$A = \operatorname{coker}(0 \to A) \to \ker(B \to 0) = B$$

is an isomorphism.

□

DEFINITION 10.8. *Let $\mathcal{A}, \mathcal{B}$ be semiadditive categories. Then a functor $F : \mathcal{A} \to \mathcal{B}$ is called additive if it preserves products (equivalently coproducts). If $\mathcal{A}$ and $\mathcal{B}$ are* *abelian then F is called left exact if it is additive and for each morphism $f : A \to B$ in $\mathcal{A}$ the map*

$$F(\ker(f)) \to \ker(F(f))$$

is an isomorphism. Similarly we define right exact using cokernels. A functor is called exact if it is additive and both left and right exact.

We now want to talk about exact sequences in an abelian category:

DEFINITION 10.9. *Let $\mathcal{A}$ be an abelian category. A cochain complex in A is a sequence of objects*

$$\ldots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \ldots$$

such that all composites $d^n \circ d^{n-1}$ are 0.

LEMMA 10.10. *For a sequence $A \to B \to C$ with composite 0, we have*

$$\begin{aligned} &\ker(\operatorname{coker}(A \to B) \to C) \\ \cong &\operatorname{coker}(A \to \ker(B \to C)) \\ \cong &\operatorname{coker}(\operatorname{im}(A \to B) \to \ker(B \to C)) \end{aligned}$$

PROOF. A map $\operatorname{coker}(A \to \ker(B \to C)) \to \ker(\operatorname{coker}(A \to B) \to C)$ is, by universal properties, given by a map $\ker(B \to C) \to \operatorname{coker}(A \to B)$ which is zero when precomposed with the map from A or postcomposed with the map to C. We have such a map, given by $\ker(B \to C) \to B \to \operatorname{coker}(A \to B)$.
In the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \operatorname{coker}(A \to B) \arrow[d] \\ C \arrow[r] & C \end{tikzcd}$$

we may take vertical and then horizontal kernels or horizontal and then vertical kernels, to see that

$$\ker(B \to C) \to \ker(\operatorname{coker}(A \to B) \to C)$$

has kernel $\operatorname{im}(A \to B)$. This shows that we have a canonical monomorphism

$$\operatorname{coker}(\operatorname{im}(A \to B) \to \ker(B \to C)) \to \ker(\operatorname{coker}(A \to B) \to C)),$$

and since $A \to \operatorname{im}(A \to B)$ is epi, we may also write this as

$$\operatorname{coker}(A \to \ker(B \to C)) \to \ker(\operatorname{coker}(A \to B) \to C)).$$

This means that our canonical morphism is mono. By working in the opposite abelian category, we also get that it is epi, so it is an isomorphism as desired. $\square$

DEFINITION 10.11. *We write*

$$H^n(C) := \ker(\operatorname{coker}(C^{n-1} \to C^n) \to C^{n+1})$$

DEFINITION 10.12. *We call a sequence $A \to B \to C$ exact in the middle if its H^1 vanishes when we view it as cochain complex, i.e. if $\ker(\operatorname{coker}(A \to B) \to C) = 0$, or equivalently $\operatorname{im}(A \to B) = \ker(B \to C)$. We call a longer sequence exact if all its parts are exact. The special case of an exact sequence*

$$0 \to A \to B \to C \to 0$$

is called a short exact sequence.

The following lemma is useful for later:

LEMMA 10.13. (1) *For a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

consider the sequence $0 \to A \to B \oplus C \to D \to 0$. *This is left exact if and only if the diagram is a pullback, right exact if and only if it is a pushout, exact if and only if it is both.*

(2) *Pullbacks of epimorphisms are epimorphisms, pushouts of monomorphisms are monomorphisms.*

PROOF. Observe that a short sequence is left exact if and only if the canonical map to the kernel is epi and mono, hence iso. So the first statement follows by comparing universal properties, analogously for the dual statement.
For the second statement, if

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pullback diagram with $C \to D$ epi, then by the above,

$$0 \to A \to B \oplus C \to D \to 0$$

is left exact. But $B \oplus C \to D$ is also epi since $C \to D$ is, so the diagram is also a pushout. By universal properties, we see that $\operatorname{coker}(A \to B) \cong \operatorname{coker}(C \to D) \cong 0$, hence $A \to B$ is epi. □

DEFINITION 10.14. *An abelian category is said to have enough injectives if for every object A there exists a monomorphism $A \to I$ into an injective object.*

Now note that δ-functors, initial δ-functors and effaceable functors make sense in the generality of abelian categories (replacing $\mathrm{Shv}(X, \mathrm{Ab})$ and Ab). Also the claim that effaceable functors are initial works in this generality.

DEFINITION 10.15. *Let C^* and D^* be cochain complexes.*

(1) *A chain map $f : C^* \to D^*$ is a collection of maps $f^n : C^n \to D^n$ which commutes with the differentials d^n.*
(2) *A chain homotopy h between two chain maps $f, g : C^* \to D^*$ is given by a collection of maps $h^n : C^n \to D^{n-1}$ with*

$$d^{n-1}h^n + h^{n+1}d^n = f^n - g^n.$$

If there exists a chain homotopy between f, g, we call f, g chain homotopic. Being chain homotopic is clearly an equivalence relation (for transitivity, add the chain homotopies).

DEFINITION 10.16. *We define $\mathrm{Ch}(\mathcal{A})$ to be the category of cochain complexes in $\mathcal{A}$ with morphisms given by chain maps and $\mathcal{K}(\mathcal{A})$ as the category of cochain complexes with morphisms given by chain homotopy classes of chain maps.*

Note that, while $\mathrm{Ch}(\mathcal{A})$ is an abelian category, $\mathcal{K}(\mathcal{A})$ is generally not!
The significance of working up to chain homotopy is in the following lemma:

LEMMA 10.17. *Cohomology defines a functor $H^n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$, which factors through $\mathcal{K}(\mathcal{A})$.*

PROOF. Clearly, cohomology is functorial, so we need to check that two chain homotopic chain maps f, g induce the same map on $H^n(C) \to H^n(D)$. Indeed, on $\ker(d^n)$, $f^n - g^n$ agrees with $d^{n-1} \circ h^n$, i.e. becomes zero in $\mathrm{coker}(D^{n-1} \to \ker(D^n \to D^{n+1}))$. □

DEFINITION 10.18. *Let $\mathcal{A}$ be an abelian category. An* injective resolution *of an object $A \in \mathcal{A}$ is given by a cochain complex*

$$I^0 \to I^1 \to I^2 \to \dots$$

of injective objects of $\mathcal{A}$, together with a map $A \to I^0$ such that the sequence

$$0 \to A \to I^0 \to I^1 \to \dots$$

is exact.

There are other ways to say this, for example an injective resolution of A can also be described as a cochain complex as above with $H^n(I) \cong 0$ for $n > 0$ and a chosen isomorphism $A \to H^0(I)$. We don't think of A as part of the cochain complex I^* (but of course the map $A \to I^0$ is part of the data of an injective resolution).

LEMMA 10.19. *If $\mathcal{A}$ has enough injectives, every object A admits an injective resolution.*

PROOF. We inductively construct objects A^i and I^i by setting $A^0 = A$, and then letting $A^i \to I^i$ be a monomorphism into an injective, and $A^{i+1} = \mathrm{coker}(A^i \to I^i)$. Then we claim the complex

$$I^0 \to I^1 \to \dots$$

is an injective resolution of A. Indeed, the kernel of $I^0 \to I^1$ agrees with the kernel of $I^0 \to A^1$, since $A^1 \to I^1$ is a monomorphism, and thus with A^0, since $A^0 \to I^0$ was a monomorphism. By the same argument, we have

$$\ker(I^i \to I^{i+1}) = A^i = \mathrm{coker}(A^{i-1} \to I^i) = \mathrm{im}(I^{i-1} \to I^i),$$

so the sequence is exact in all higher degrees. □

THEOREM 10.20 (Fundamental Lemma of homological algebra). *Let C be any cochain complex whose cohomology is concentrated in degree 0 (i.e. $H^i(C) = 0$ for $i \neq 0$), and let I be a complex of injectives concentrated in degrees ≥ 0. (For example, both could be injective resolutions.) Then H^0 induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(C, I) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(H^0(C), H^0(I)).$$

PROOF. We first show injectivity. Since the map is a homomorphism, it suffices to check that if $f : C \to I$ induces the zero map on H^0 (and hence all cohomology), we find a chain homotopy between f and 0.
Suppose we have inductively found $h^i : C_i \to I_{i-1}$ for $i \leq n$, with $d^{i-1}h^i + h^{i+1}d^i = f_i$ for $i \leq n-1$. (We start with $n = 0$, which is trivial.) We now have to find $h^{n+1} : C_{n+1} \to I_n$ with

$$h^{n+1}d^n = f^n - d^{n-1}h^n. \tag{16}$$

By construction, the right-hand map, precomposed with $d^{n-1} : C^{n-1} \to C^n$, equals:

$$(f^n - d^{n-1}h^n)d^{n-1} = f^n d^{n-1} - d^{n-1}(d^{n-2}h^{n-1} + h^n d^{n-1}) = f^n d^{n-1} - d^{n-1}f^{n-1} = 0$$

So the right hand map factors through a map $\operatorname{coker}(d^{n-1}) \to I^n$. Furthermore, the map vanishes on $H^{n-1}(C) = \ker(\operatorname{coker}(d^{n-1}) \to C^{n+1})$. This is clear for $n \geq 2$, and for $n = 1$ it comes from the fact that the map from $H^0(C) \to \ker(I^0 \to I^1) = H^0(I)$ is precisely the effect of our f on cohomology, which is zero by assumption. So the right hand map $f^n - d^{n-1}h^n$ even factors through $\operatorname{im}(C^n \to C^{n+1})$. By injectivity of I_n, we may extend it over the monomorphism to C^{n+1}, the resulting map $h^{n+1} : C^{n+1} \to I^n$ solves (16).
We now show surjectivity. For this, we first observe that we may extend a homomorphism $H^0(C) \to H^0(I) = \ker(I^0 \to I^1)$ over the monomorphism $H^0(C) \to \operatorname{coker}(C^{-1} \to C^0)$ to obtain a map $f^0 : C^0 \to I^0$. We now assume we have constructed $f^i : C^i \to I^i$ for $i \leq n$ with $d^{i-1}f^{i-1} = f^i d^{i-1}$ for $i \leq n$. We need to find a f^{n+1} with $f^{n+1}d^n = d^n f^n$. The right hand side vanishes when precomposed with d^{n-1}, so factors over a map $\operatorname{coker}(d^{n-1}) \to I^{n+1}$. Since it even vanishes on $H^n(C)$ (for $n \geq 1$ this is clear, for $n = 0$ we have constructed f^0 to take the kernel of d^0 to the kernel of d^0), it even factors over $\operatorname{im}(C^n \to C^{n+1})$. Using injectivity of I^{n+1}, we may extend over all of C^{n+1} to obtain our f^{n+1}. □

Corollary 10.21. *Let $\mathcal{A}$ have enough injectives, and let $\mathcal{C} \subseteq \mathcal{K}(\mathcal{A})$ denote the full subcategory of cochain complexes which are injective resolutions, i.e. levelwise injective, concentrated in degrees ≥ 0, and with homology concentrated in degree 0. Then the functor H_0 induces an equivalence of categories $\mathcal{C} \cong \mathcal{A}$. In particular, the inverse equivalence gives a well-defined functor*

$$I^* : \mathcal{A} \to \mathcal{K}(\mathcal{A})$$

which takes each object $A \in \mathcal{A}$ to an injective resolution of A.

We are now able to define the functors RF^i from 9.13, simply as the composite

$$\mathcal{A} \xrightarrow{I^*} \mathcal{K}(\mathcal{A}) \xrightarrow{F} \mathcal{K}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}$$

where the first functor is our injective resolution functor, and the second functor levelwise applies F, which is well-defined since F is additive and hence in particular preserves chain homotopies. In order to make these into a δ-functor, we need however one more ingredient, namely the long exact sequence which is part of the structure of a δ-functor (i.e. the connecting homomorphisms δ).

Definition 10.22. *Let $f : C \to D$ be a chain map. We define a new chain complex* $\operatorname{Cone}(f)$ *by*

$$\operatorname{Cone}(f)^n = D^n \oplus C^{n+1},$$

with $d^n_{\operatorname{Cone}(f)} : D^n \oplus C^{n+1} \to D^{n+1} \oplus C^{n+2}$ given by

$$\begin{pmatrix} d^n_D & f^{n+1} \\ 0 & -d^{n+1}_C \end{pmatrix}$$

It is not hard to see that this is indeed a chain complex. Moreover we have a map $D \to \operatorname{Cone}(f)$ given by the inclusion and a map $\operatorname{Cone}(f) \to C[1]$ given by the projection where $C[1]$ is the chain complex C shifted up by one, i.e. $C[1]^n = C^{n+1}$. We now claim that the sequence

$$\to (\operatorname{Cone} f)[-1] \to C \to D \to \operatorname{Cone}(f) \to C[1] \to D[1] \to \ldots$$

induces an exact sequence in cohomology. In order to prove this we need to check exactness of a sequence. In a category of modules, we may do this by a simple diagram chase, using elements. This is not available in a general abelian category, but we may work with generalized elements, i.e. homomorphisms $Z \to A$.

LEMMA 10.23. *A sequence*

$$A \to B \to C$$

in an abelian category, is exact if and only if for any $Z \to B$ with composite $Z \to C$ being 0, there exists an epimorphism $\widetilde{Z} \to Z$ such that $\widetilde{Z} \to B$ factors through A, i.e.

$$\begin{tikzcd} \widetilde{Z} \arrow[r, two heads] \arrow[d, dashed] & Z \arrow[d] \arrow[dr, "0"] & \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

PROOF. If $A \to B \to C$ is exact, $A \to \ker(B \to C)$ is an epimorphism. We may now take $\widetilde{Z}$ to be the pullback along $Z \to \ker(B \to C)$, and use that pullbacks of epimorphisms are epimorphisms.
Conversely, take $Z = \ker(B \to C)$. Then $\widetilde{Z} \to Z \to \ker(B \to C)$ is an epimorphism, and so $A \to \ker(B \to C)$ is, too. □

LEMMA 10.24. *We have natural long exact sequences*

$$\ldots \to H^n(C) \to H^n(D) \to H^n(\mathrm{Cone}(f)) \to H^{n+1}(C) \to \ldots$$

where the first map is induced by f, the second comes from the canonical inclusion of complexes, and the third from the canonical projection of complexes.

PROOF. We need to show exactness. This can be done by a diagram chase with generalized elements, we demonstrate this for exactness at $H^n(C)$, the others work similarly.
Let $Z \to H^n(C)$ be a map such that the composite $Z \to H^n(C) \to H^n(D)$ vanishes. We need to find an epimorphism $\widetilde{Z} \to Z$ such that $\widetilde{Z} \to H^n(C)$ lifts over $H^{n-1}(\mathrm{Cone}(f))$. Indeed, we first choose an epimorphism $Z' \twoheadrightarrow Z$ such that $Z \to H^n(C)$ lifts to a map $Z' \to \ker(d_C^n)$ (e.g. take the pullbacks). Now since $Z \to H^n(C) \to H^n(D)$ vanishes, it means that we find $\widetilde{Z} \twoheadrightarrow Z'$ such that $Z' \to \ker(d_D^n)$ lifts over $D^{n-1} \to \ker(d_D^n)$. This gives us a map (with appropriate sign)

$$\widetilde{Z} \to C^n \oplus D^{n-1}$$

in the kernel of $\delta^{n-1}_{\mathrm{Cone}(f)}$, lifting the map $\widetilde{Z} \to Z \to C^n$. □

Naturality here means natural in the pair $C \to D$. We may view these pairs as a category, which we call the *arrow category*: An object of $\mathrm{Ar}(\mathcal{C})$ for any category $\mathcal{C}$ consists of a morphism $x \to y$ in $\mathcal{C}$, and a morphism between such arrows is a commutative square. The natural long exact sequence above is hence a functor from $\mathrm{Ar}(\mathcal{K}(\mathcal{A}))$ into a category of long exact sequences in $\mathcal{A}$. Note that $\mathrm{Ch}(\mathrm{Ar}(\mathcal{A})) = \mathrm{Ar}(\mathrm{Ch}(\mathcal{A}))$. There is also a canonical functor

$$\mathcal{K}(\mathrm{Ar}(\mathcal{A})) \to \mathrm{Ar}(\mathcal{K}(\mathcal{A}))\ .$$

However, while this functor is essentially surjective it is in general not an equivalence.

EXAMPLE 10.25. Let $\mathcal{A} = \mathrm{Ab}$ be the category of abelian groups and consider the object

$$\mathbb{Z}[0] \to \mathrm{Cone}(\mathrm{id}_{\mathbb{Z}[0]})$$

of $\mathcal{K}(\mathrm{Ar}(\mathcal{A}))$.Here $\mathbb{Z}[0]$ is the chain complex with $\mathbb{Z}$ in degree 0 and Cone(id) is the cochain complex

$$0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \to 0 \to ...$$

with $\mathbb{Z}$'s concentrated in degree 0 and -1. Clearly Cone(id) is chain homotopy equivalent to 0, thus in $\mathrm{Ar}(\mathcal{K}(\mathcal{A}))$ this object is equivalent to $\mathbb{Z}[0] \to 0$. On the other hand in $\mathcal{K}(\mathrm{Ar}(\mathcal{A}))$ there is no non-trivial map in $\mathrm{Ar}(\mathcal{A})$ from $\mathbb{Z}[0] \to 0$ to $\mathbb{Z}[0] \to \mathrm{Cone}(\mathrm{id}_{\mathbb{Z}[0]})$. In particular they cannot be isomorphic in $\mathcal{K}(\mathrm{Ar}(\mathcal{A}))$.

It is also not hard to see that if I and J are injective in $\mathcal{A}$, then $I \oplus J \to J$ is an injective object in $\mathrm{Ar}(\mathcal{A})$.[5] In particular if $\mathcal{A}$ has enough injectives then $\mathrm{Ar}(\mathcal{A})$ does, too: for $M \to N$ we simply choose monos $M \to I$ and $N \to J$ and then pick the resulting map $(M \to N) \to (I \oplus J \to J)$. By choosing an injective resolution in $\mathrm{Ar}(\mathcal{A})$ we get a functor

$$\mathrm{Ar}(\mathcal{A}) \xrightarrow{I^*} \mathcal{K}(\mathrm{Ar}(\mathcal{A}))$$

which lifts the functor

$$\mathrm{Ar}(\mathcal{A}) \xrightarrow{\mathrm{Ar}(I^*)} \mathrm{Ar}(\mathcal{K}(\mathcal{A})).$$

The construction Cone is naturally a functor

$$\mathrm{Ch}(\mathrm{Ar}(\mathcal{A})) = \mathrm{Ar}(\mathrm{Ch}(\mathcal{A})) \to \mathrm{Ch}(\mathcal{A})$$

LEMMA 10.26. *If we let* $\mathrm{Ar}(\mathcal{A})^{\mathrm{mono}}$ *denote the full subcategory on arrows which are monomorphisms, then we have a natural isomorphism between the composite functors*

$$\begin{tikzcd} \mathrm{Ar}(\mathcal{A})^{\mathrm{mono}} \arrow[r, "I^*"] \arrow[d, "\mathrm{coker}"] & \mathcal{K}(\mathrm{Ar}(\mathcal{A})) \arrow[d, "\mathrm{Cone}"] \\ \mathcal{A} \arrow[r, "I^*"] & \mathcal{K}(\mathcal{A}) \end{tikzcd}$$

Similarly, we have natural isomorphisms

$$\begin{tikzcd} \mathrm{Ar}(\mathcal{A}) \arrow[r, "I^*"] \arrow[d, "s \text{ or } t"] & \mathcal{K}(\mathrm{Ar}(\mathcal{A})) \arrow[d, "s \text{ or } t"] \\ \mathcal{A} \arrow[r, "I^*"] & \mathcal{K}(\mathcal{A}) \end{tikzcd}$$

PROOF. We first do the second (and third) diagrams. Both composites land in the full subcategory of injective resolutions in $\mathcal{K}(\mathcal{A})$. This is clear for the lower left composite, and for the upper right one we need that for an injective object of $\mathrm{Ar}(\mathcal{A})$, both source and target are injective. This is easy to see. Since H^0 gives an equivalence between this full subcategory of injective resolutions and $\mathcal{A}$, we may check the statement after postcomposition, but then it is obvious.

In the first diagram, both functors again take values in the full subcategory of $\mathcal{K}(\mathcal{A})$ of injective resolutions. This is clear for the lower left composite, and for the other we observe that Cone of a map between injective resolutions consists of injectives, is concentrated in degrees ≥ 0, and if we started with a monomorphism arrow $A \to B$, the long exact sequence for

$$\mathrm{Cone}(I^*(A \to B))$$

[5] In fact, every injective object of $\mathrm{Ar}(\mathcal{A})$ has this form. An arbitrary injective object can be embedded into one of those as we will argue above. Thus it ends up being a retract. But this implies that it is split surjective with injective target and kernel, thus of the claimed form.

shows that $\mathrm{Cone}(I^*(A \to B))$ has cohomology concentrated in degree 0 (given by the cokernel). So we may again check commutativity of the diagram by postcomposing with H^0. In that case it exactly reduces to the observation that $H^0(\mathrm{Cone}(I^*(A \to B))) \cong \mathrm{coker}(A \to B)$. $\square$

Now we can finally come back to give a proof of Proposition 9.13 and thereby Theorem 9.2.

PROPOSITION 10.27. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ an additive, left-exact functor, and assume that $\mathcal{A}$ has enough injectives. Then there exists an effaceable δ-functor (R^iF, δ^i) extending F.*

PROOF. We define RF^i as the composite

$$\mathcal{A} \xrightarrow{I^*} \mathcal{K}(\mathcal{A}) \xrightarrow{\mathcal{K}(F)} \mathcal{K}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$

For an injective object I, we have $I^*(I) \cong I[0]$ in $\mathcal{K}(\mathcal{A})$, where $I[0]$ denotes the cochain complex with I concentrated in degree 0. (Since both are injective resolutions, and H^0 takes them to the same object.) So we directly see that $RF^i(I) = 0$ for $i > 0$.
All that's left to do is to endow the RF^i with the structure of a δ-functor, i.e. naturally associate to each short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

a long exact sequence. Consider the composite

$$\mathrm{Ar}(\mathcal{A})^{\mathrm{mono}} \xrightarrow{I^*} \mathcal{K}(\mathrm{Ar}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathrm{Ar}(\mathcal{B})) \to \text{long exact sequences}$$

which takes $A_1 \hookrightarrow A_2$ to the long exact sequence associated to $F(I^*(A_1 \hookrightarrow A_2))$. By Lemma 10.26, we have natural equivalences

$$FI^*(A_1 \hookrightarrow A_2) \cong F(I^*(A_1) \to I^*(A_2)) = (FI^*(A_1) \to FI^*(A_2)),$$

$$\mathrm{Cone}\, FI^*(A_1 \hookrightarrow A_2)) \cong F\,\mathrm{Cone}(I^*(A_1 \hookrightarrow A_2)) \cong FI^*(\mathrm{coker}(A_1 \to A_2)) \cong FI^*(A_3),$$

since applying F preserves cones (F is additive). So the long exact sequence we have functorially associated to each short exact sequence indeed has terms given by

$$\ldots \to RF^i(A_1) \to RF^i(A_2) \to RF^i(A_3) \to RF^{i+1}(A_1) \to \ldots.$$

In particular, we get natural transformations δ^i as desired.
Finally, we need to check $RF^0 = F$. From the fundamental theorem, we have a natural transformation $A[0] \to I^*(A)$, so applying F and taking cohomology, we see that we have a natural map $F(A) \to RF^0(A)$. To check that it is an iso, we take an injective resolution of A and observe that $0 \to A \to I^0 \to I^1$ is exact, so $0 \to F(A) \to F(I^0) \to F(I^1)$ is, too, and so the canonical map $F(A) \to \ker(F(I^0) \to F(I^1)) \cong H^0(I^*(A))$ is an isomorphism. $\square$

Finally we note that we can now derived functors. Given a left exact functor

$$F : \mathcal{A} \to \mathcal{B}$$

we can define R^iF, the i-th derived functor, as the initial δ-functor extending F (if such an initial extension exists). We have seen that, if $\mathcal{A}$ has enough injective objects it does indeed exist and is given by

$$R^iF(A) := H^i(F(I^*)) \ .$$

Then sheaf cohomology is the derived functor of global sections.

11. Cohomology and $\mathcal{O}_X$-modules

We have defined sheaf cohomology in the last sections. Now assume that X is a scheme and $\mathcal{M}$ is an $\mathcal{O}_X$-module (e.g. a quasi-coherent sheaf). Now we define sheaf cohomology of X with values in $\mathcal{M}$ by simply ignoring the fact that $\mathcal{M}$ is an $\mathcal{O}_X$-module, i.e. take the derived functor of

$$\Gamma : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$$

as before. We can also take the derived functor of

$$\Gamma : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(X)}$$

or the derived functor of

$$\Gamma : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}$$

obtained by further forgetting the module structure over the global sections $\mathcal{O}_X(X)$.

Proposition 11.1. *All possible derived functors $R^i\Gamma$ exist and agree (under the obvious forgetful functors).*

We will eventually prove this result here. But we first note that all the categories have enough injective objects. For sheaves of abelian groups we have already seen this. For $\mathrm{Mod}(\mathcal{O}_X)$ it follows formally from this result by coinducing injective objects, similar to the argument given in Corollary 9.7: for a given $\mathcal{O}_X$-module $\mathcal{M}$ we choose a mono to an injective sheaf of abelian groups $\mathcal{M} \to \mathcal{I}$. Then we get an injective sheaf of $\mathcal{O}_X$-modules as $\underline{\mathrm{Hom}}(\mathcal{O}_X, \mathcal{I})$. Injecitivity can be seen similar to the argument in Corollary 9.7 and the morphism $\mathcal{M} \to \underline{\mathrm{Hom}}(\mathcal{O}_X, \mathcal{I})$ is clearly injective. Thus the existence part for the derived functors is clear.
The fact that the derived functors of $\Gamma : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(X)}$ and $\Gamma : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}$ agree is also immediate, since the forgetful functor $\mathrm{Mod}_{\mathcal{O}_X(X)} \to \mathrm{Ab}$ is exact and therefore preserves cohomology of complexes. Finally to compare it to the derived functor of $\Gamma : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$, i.e. sheaf cohomology, it suffices to show that

$$\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Shv}(X, \mathrm{Ab}) \xrightarrow{H^i(X,-)} \mathrm{Ab}$$

for $i > 0$ is still an effacable functor, i.e. vanishes on injective $\mathcal{O}_X$-modules. This is true as we will proof now, but it is not formal since general injective $\mathcal{O}_X$-modules are not injective as sheaves of abelian groups. Therefore we need to establish some terminology.

Definition 11.2. *A sheaf $\mathcal{F}$ on a space X is called* flasque *if for all inclusions $U \subseteq V \subseteq X$ the induced map $F(V) \to F(U)$ is surjective.*

Lemma 11.3. *If a sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is injective (as a sheaf of $\mathcal{O}_X$-modules), then it is flasque.*

Proof. For a given open set $U \subseteq X$ we consider the sheafification of the presheaf of $\mathcal{O}_X$-modules

$$i_U(V) = \begin{cases} \mathcal{O}_X(V) & \text{if } V \subseteq U \\ 0 & \text{else} \end{cases}$$

We have that for any sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ that

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(i_U, M) = M(U) \ .$$

by a version of the Yoneda lemma. Moreover we have that $(i_U)_x = \mathcal{O}_x$ for $x \in U$ and $(i_U)_x = 0$ else.

Now for an inclusion $U \subseteq V$ the induced map $i_U \to i_V$ is a monomorphism and thus we can conclude that

$$M(V) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(i_V, M) \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(i_U, M) = M(U)$$

is surjective. $\square$

Lemma 11.4. *If $\mathcal{F}$ is a flasque sheaf of abelian groups and*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is a short exact sequence. Then the induced map

$$\mathcal{G}(X) \to \mathcal{H}(X)$$

is surjective.

Proof. Let $s \in \mathcal{H}(X)$. Using Zorn's Lemma we pick a maximal pair (U, s_1) where s_1 is a lift of s in $\mathcal{G}(U)$. If $U \neq X$ then pick a point $x \notin X$ and a lift of s in a neighborhood V, i.e. an element $s_2 \in \mathcal{H}(V)$. This is possible since the map is surjective on stalks. Now we consider

$$s_1|_{U \cap V} - s_2|_{U \cap V} \in \mathcal{G}(U \cap V)$$

This element actually lies in $\mathcal{F}(U \cap V)$ since ist image in $\mathcal{H}(U \cap V)$ vanishes. This using that $\mathcal{F}$ is flabby we can extend it to an element

$$t \in \mathcal{F}(V) \ .$$

Then $s_2 + t \in \mathcal{G}(V)$ is still a lift of s and on $U \cap V$ agrees with s_1. Thus we can glue the two sections s_1 and $s_2 + t$ to a section $\mathcal{G}(U \cup V)$ lifting s. This contradicts the maximality of U. $\square$

Lemma 11.5. *Let $\mathcal{F}$ be a flasque sheaf of abelian groups. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Choose a mono $\mathcal{F} \to \mathcal{I}$ and consider $\mathcal{G} := \mathcal{F}/\mathcal{I}$. Then we have from the long exact sequence that

$$H^1(X, \mathcal{F}) \cong H^0(X, \mathcal{G})/H^0(X, \mathcal{I})$$

and

$$H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$$

for $i > 1$. The first vanishes by the previous lemma. We finish the argument inductively by the claim that $\mathcal{G}$ is also flasque. This follows since $\mathcal{F}$ and $\mathcal{I}$ are flasque (the latter by an argument similar to 11.3). But then for $U \subseteq V$ we have a commutative square

$$\begin{tikzcd}
\mathcal{I}(V) \arrow[r] \arrow[d, two heads] & \mathcal{G}(V) \arrow[d] \\
\mathcal{I}(U) \arrow[r, two heads] & \mathcal{G}(U)
\end{tikzcd}$$

in which the lower horizontal morphisms is surjective by the fact that $\mathcal{F}|_U$ is an injective sheaf and the previous lemma applied to

$$0 \to \mathcal{F}|_U \to \mathcal{I}|_U \to \mathcal{G}|_U \to 0 \ .$$

It follows that $\mathcal{G}(V) \to \mathcal{G}(U)$ is surjective. $\square$

REMARK 11.6. If $\mathcal{M}$ is quasi-coherent, then one can also take the derived functor of either

$$\Gamma : \mathrm{QCoh}(X) \to \mathrm{Mod}_{\mathcal{O}_X(X)} \qquad \text{or} \qquad \Gamma : \mathrm{QCoh}(X) \to \mathrm{Ab} \ .$$

The fact that one can do this follows since one can show that $\mathrm{QCoh}(X)$ has enough injectives, which is a Theorem of Gabber. However it turns out that it is generally not true, that these derived functors also agree.

THEOREM 11.7. *Let $X = \mathrm{Spec}(R)$ be affine and $\mathcal{M}$ be quasi-coherent. Then $H^i(X, \mathcal{M}) = 0$ for $i > 0$.*

In order to prove this result, we shall introduce Čech cohomology. Let X be a topological space and $\{U_i\}_{i\in I}$ be an open covering of X. We set for $i_0, ..., i_n \in I$

$$U_{i_0,...,i_n} := U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_n} \ .$$

We will moreover assume that I is equipped with a total ordering.

DEFINITION 11.8. *For a sheaf $\mathcal{F}$ of abelian groups we define the Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$ as the cochain complex*

$$\check{C}(\mathcal{U}, \mathcal{F}) = \left(\prod_{i_0 \in I} \mathcal{F}(U_i) \xrightarrow{d} \prod_{i_0 < i_1} \mathcal{F}(U_{i_0,i_1}) \xrightarrow{d} \prod_{i_0<i_1<i_2} \mathcal{F}(U_{i_0,i_1,i_2}) \xrightarrow{d} ... \right)$$

where the differentials are given as the alternating sum of restrictions. Concretely for a family $s = (s_{i_0,...,i_{n-1}})_{i_0,...,i_{n-1}} \in \prod_{i_0<...<i_{n-1}} \mathcal{F}(U_{i_0,...,i_{n-1}})$ we have

$$d(s)_{i_0,...,i_n} = \sum_{k=0}^{n} (-1)^k s_{i_0,...,\hat{i}_k,...,i_n}|_{U_{i_0,...,i_n}}$$

where $\hat{i}_k$ indicates to skip the k-th entry.

One can easily check that this is indeed a cochain complex:

$$\begin{aligned} d(d(s))_{i_0,...,i_n} &= \sum_{k=0}^{n} (-1)^k ds_{i_0,...,\hat{i}_k,...,i_n}|_{U_{i_0,...,i_n}} \\ &= \sum_{k=0}^{n} (-1)^k \left(\sum_{l<k}^{n-1} (-1)^l s_{i_0,...,\hat{i}_l,..,\hat{i}_k,...,i_n}|_{U_{i_0,...,i_n}} + \sum_{l\geq k}^{n-1} (-1)^l s_{i_0,...,\hat{i}_k,...,\hat{i}_{l+1},...,i_n}|_{U_{i_0,...,i_n}} \right) \\ &= \end{aligned}$$

For $a < b$ the term $s_{i_0,...,\hat{a},..,\hat{b},...,i_n}|_{U_{i_0,...,i_n}}$ occurs twice in this sum at indices $l = a, k = b$ and $l = b-1, k = a$. But the signs are different so that they cancel and $d^2(s) = 0$.

DEFINITION 11.9. *We define the Čech-cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$ as the cohomology*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = H^n(\check{C}(\mathcal{U}, \mathcal{F})).$$

EXAMPLE 11.10. For $n = 0$ we have

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker \left(\prod_{i_0} \mathcal{F}(U_{i_0}) \to \prod_{i_0<i_1} \mathcal{F}(U_{i_0,i_1}) \right) \ .$$

By the sheaf property for $\mathcal{F}$ this is isomorphic to $\mathcal{F}(X)$, so that $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

EXAMPLE 11.11. If $\mathcal{U}$ consists of two open sets U and V then the Čech complex takes the form

$$F(U) \times F(V) \xrightarrow{\mathrm{res}_2 - \mathrm{res}_1} F(U \cap V) \to 0 \to 0 \to ...$$

In particular the cohomology in degree 1 is given by elements in $F(U \cap V)$ modulo differences $s_2 - s_1$ of sections $s_1 \in \mathcal{F}(V)$ and $s_2 \in \mathcal{F}(U)$.

EXAMPLE 11.12. If $\mathcal{U}$ consists of three open sets U, V, Wthen the Čech complex takes the form

$$F(U)\times F(V)\times F(W) \xrightarrow{(\mathrm{res}_2 - \mathrm{res}_1)} F(U\cap V)\times F(U\cap W)\times F(V\cap W) \to F(U\cap V\cap W) \to 0 \to 0 \to ...$$

LEMMA 11.13. *Assume that there exists an $j \in I$ such that $U_j = X$. Then*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \begin{cases} \mathcal{F}(X) & n = 0 \\ 0 & \textit{else} \end{cases}$$

PROOF. We assume for simplicity that j is minimal and call it 1 (otherwise permute the order of I). We simply expand the Čech complex

$$\check{C}(\mathcal{U}, \mathcal{F}) = \left(\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \xrightarrow{d} \prod_{i_0 < i_1} \mathcal{F}(U_{i_0, i_1}) \xrightarrow{d} \prod_{i_0 < i_1 < i_2} \mathcal{F}(U_{i_0, i_1, i_2}) \xrightarrow{d} ... \right)$$

and compare it to the cochain complex

$$\mathcal{F}(X)[0] = (\mathcal{F}(X) \to 0 \to 0 \to ...)$$

There is a chain map

$$\mathcal{F}(X)[0] \to \check{C}(\mathcal{U}, \mathcal{F})$$

given by restriction along $U_{i_0} \to X$. But there is also a map

$$\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(X)[0]$$

given by projection to the first factor. The composition $\mathcal{F}(X)[0] \to \mathcal{F}(X)[0]$ is clearly the identity. The composition

$$\check{C}(\mathcal{U}, \mathcal{F}) \to \check{C}(\mathcal{U}, \mathcal{F})$$

is not the identity, but we claim that it is chain homotopy equivalent to the identity by means of the chain homotopy

$$h : \prod_{i_0 < ... < i_n} \mathcal{F}(U_{i_0, ..., i_n}) \to \prod_{i_0 < ... < i_{n-1}} \mathcal{F}(U_{i_0, ..., i_{n-1}})$$

which is given in the $i_0 < ... < i_{n-1}$-coordinate by the projection to the $1 < i_0 < ... < i_{n-1}$ factor if $i_0 \neq 1$ and by 0 else. One then verifies that this is indeed the required chain homotopy. □

LEMMA 11.14. *Assume that $\mathcal{M}$ is a quasi-coherent sheaf on $X = \mathrm{Spec}(R)$ and $(U_i = D(f_i))_{i \in I}$ is a (finite) cover by principal opens. Then*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \begin{cases} \mathcal{F}(X) & n = 0 \\ 0 & \textit{else} \end{cases}$$

Proof. We have to show that the complex

$$\mathcal{M}(X) \to \bigoplus_{i_0 \in I} \mathcal{M}(U_{i_0}) \xrightarrow{d} \bigoplus_{i_0<i_1} \mathcal{M}(U_{i_0,i_1}) \xrightarrow{d} \bigoplus_{i_0<i_1<i_2} \mathcal{M}(U_{i_0,i_1,i_2}) \xrightarrow{d} ...$$

is exact. This is a complex of R-modules and thus we can check exactness by faithfully flat descent after basechange along $R \to R[f_j^{-1}]$ for each j. But then the resulting complex is the Cech complex of the cover $(U_j \cap U_i)_{i\in I}$ which has the property that one of the opens agrees with the full set U_j. Thus by the previous lemma the claim follows. □

Lemma 11.15. *Assume that $\mathcal{F}$ is injective as a sheaf or as an $\mathcal{O}_X$-module. Then*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \begin{cases} \mathcal{F}(X) & n = 0 \\ 0 & else \end{cases}$$

In particular this agrees with sheaf cohomology.

Proof. We cover the case of an injective $\mathcal{O}_X$-module. The case of an injective sheaf works similar. Recall for an open $U \subseteq X$ the sheaf i_U from the proof of Lemma 11.3 given as the sheafification of

$$i_U(V) = \begin{cases} \mathcal{O}_X(V) & \text{if } V \subseteq U \\ 0 & \text{else} \end{cases}$$

with

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(i_U, \mathcal{F}) = \mathcal{F}(U) \ .$$

Now for a given cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the object corpresenting the augmented Čech complex.

$$\mathcal{F}(X) \to \prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \xrightarrow{d} \prod_{i_0<i_1} \mathcal{F}(U_{i_0,i_1}) \xrightarrow{d} \prod_{i_0<i_1<i_2} \mathcal{F}(U_{i_0,i_1,i_2}) \xrightarrow{d} ...$$

given as

$$i_X \leftarrow \bigoplus_{i_0} i_{U_{i_0}} \leftarrow \bigoplus_{i_0<i_1} i_{U_{i_0,i_1}} \leftarrow \bigoplus_{i_0<i_1<i_2} i_{U_{i_0,i_1,i_2}} \leftarrow ...$$

We claim that the latter is exact as a chain complex of sheaves. This would imply the claim since by injectivity of $\mathcal{F}$ this also implies that the augmented Čech complex is exact. For exactness of the corepresenting complex we simply observe that this can be checked on stalks for each $x \in X$. But each $x \in X$ is contained in some open U_j and we may without loss of generality replace X by U_i. But then we have that one of the open sets in the cover agrees with X and we get the exactness by an argument similar to the previous Lemma, i.e. writing down an explicit chain nullhomotopy. □

Lemma 11.16. *Let*

$$0 \to \mathcal{M} \to \mathcal{M}' \to \mathcal{M}'' \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules and $U \subseteq X$ be an open subset. Assume that each covering of U admits a refinement $\mathcal{U}$ such that $\check{H}^1(\mathcal{U}, \mathcal{M}) = 0$, then the map

$$\mathcal{M}'(U) \to \mathcal{M}''(U)$$

is surjective.

Proof. Let $s \in \mathcal{M}''(U)$. We can find a covering $\mathcal{U} = (U_i)$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{M}) = 0$ and such that there exists lifts $s_i \in \mathcal{M}'(U_i)$ of $s|_{U_i}$. Now consider

$$s_{i_0,i_1} := s_{i_0}|_{U_{i_0,i_1}} - s_{i_1}|_{U_{i_0,i_1}} .$$

This lies in $\mathcal{M}(U_{i_0,i_1})$ as one easily verifies. Moreover it defines a Čech cocycle, and thus by the vanishing of Čech cohomology a boundary, that is there are $t_i \in \mathcal{M}(U_i)$ such that $s_{i_0,i_1} = t_{i_1} - t_{i_0}$. But then $s_i - t_i \in \mathcal{M}'(U_i)$ agree on double overlaps and therefore define a lift of s in $\mathcal{M}'(U)$. □

Proof of Theorem 11.7. As in the theorem let $X = \mathrm{Spec}(R)$ and $\mathcal{M}$ quasi-coherent. Pick a monomorphism $\mathcal{M} \to \mathcal{I}$ with $\mathcal{I}$ injective (in $\mathcal{O}_X$-modules) and denote the quotient by $\mathcal{Q}$. Using the previous Lemma in combination with Lemma 11.14 we deduce that

$$0 \to \mathcal{M}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0$$

is short exact for every affine open $U \subseteq X$. Thus for any cover $\mathcal{U}$ consisting of principal opens we get a levelwise short exact sequence

$$0 \to \check{C}(\mathcal{U}, \mathcal{M}) \to \check{C}(\mathcal{U}, \mathcal{I}) \to \check{C}(\mathcal{U}, \mathcal{Q}) \to 0$$

of Čech complexes (every term is given by products of terms as above). From the long exact sequence in Čech cohomology we deduce that the higher Čech cohomology groups or all three sheaves vanish with respect to principal covers (for the first two these are Lemma 11.14 and Lemma 11.15).
Now we consider the long exact sequence in sheaf cohomology associated to the sequence $\mathcal{M} \to \mathcal{I} \to \mathcal{Q}$. We find that $H^1(X, \mathcal{M}) = 0$ by surjectivity of $\mathcal{I}(X) \to \mathcal{Q}(X)$ and vanishing of $H^1(X, \mathcal{I})$ and $H^n(X, \mathcal{M}) = H^{n-1}(X, \mathcal{Q})$.
Now we can repeat the whole argument replacing $\mathcal{M}$ by $\mathcal{Q}$ since we have only used about $\mathcal{M}$ that the higher Čech cohomology wrt principal covers vanishes and this is also satisfied for $\mathcal{Q}$ as shown above. Then induction finishes the proof. □

Recall that we would like to think of Čech cohomology as a ‘resolution’ of the space X. Now we would like to prove that one can also use this to compute sheaf cohomology. The idea is to resolve both at the same time and see what we get. Precisely what we mean is that we choose an injective resolution

$$I^0 \xrightarrow{\partial} I^1 \xrightarrow{\partial} I^2 \xrightarrow{\partial} \ldots$$

of $\mathcal{F}$ and then consider the Čech complex for each of the $\mathcal{I}^n$. This give the following commutative diagram

$$\begin{tikzcd}
\prod_{i_0\in I} \mathcal{I}^0(U_i) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1} \mathcal{I}^0(U_{i_0,i_1}) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1<i_2} \mathcal{I}^0(U_{i_0,i_1,i_2}) \arrow[r, "d"] \arrow[d, "\partial"] & \ldots \\
\prod_{i_0\in I} \mathcal{I}^1(U_i) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1} \mathcal{I}^1(U_{i_0,i_1}) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1<i_2} \mathcal{I}^1(U_{i_0,i_1,i_2}) \arrow[r, "d"] \arrow[d, "\partial"] & \ldots \\
\prod_{i_0\in I} \mathcal{I}^2(U_i) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1} \mathcal{I}^2(U_{i_0,i_1}) \arrow[r, "d"] \arrow[d, "\partial"] & \prod_{i_0<i_1<i_2} \mathcal{I}^2(U_{i_0,i_1,i_2}) \arrow[r, "d"] \arrow[d, "\partial"] & \ldots \\
\ldots & \ldots & \ldots &
\end{tikzcd}$$

called a double complex. By definition a double complex is a chain complex in chain complexes. Concretely a double complex in some abelian category $\mathcal{A}$ is given by objects $A^{i,j}$ for $i, j \in \mathbb{Z}$ with maps $\partial : A^{i,j} \to A^{i+1,j}$ and $d : A^{i,j} \to A^{i,j+1}$ such that $d\partial = \partial d$ and $d^2 = \partial^2 = 0$, i.e. all rows and all columns are cochain complexes. In our situation the double complex is concentrated in the first quadrant, i.e. $A^{i,j} = 0$ if $i < 0$ or $j < 0$. In a double complex one can take the cohomology of each row $H^n(A^{i,j}, d)$ (for fixed i and n) which still gives a square grid of objects of A. This then still has am operator ∂ that makes it into cochain complexes for each n. Then one can take the cohomology of the columns as well to get a square grid of objects in $\mathcal{A}$ (without any further maps). One can of course also do this in reverse order, i.e. first take cohomology of the columns and then of the rows. These two things are generally quite different, but we will see now that they sometimes agree.

Note that in our double complex the rows have the property that their cohomology is concentrated in degree 0 by Lemma 11.15 and since restrictions of injective sheaves to open subsets are still injective. We have the following important property.

Lemma 11.17. *Assume that a first quadrant double complex $(A^{i,j}, d, \partial)$ in $\mathcal{A}$ has the property that all rows and columns individually have cohomology concentrated in degree* 0*. Then we have natural isomorphisms*

$$H^n(H^0(A^{i,j}, d), \partial) \cong H^n(H^0(A^{i,j}, \partial), d) \ .$$

Proof. Omitted, but the idea is to define a third object, called the total complex and the total cohomology by taking the sum of diagonal entries. Then one compares both sides to this third object. □

Proposition 11.18. *Assume that for all intersections $U_{i_0,\dots,i_n}$ with $i_0 < \dots < i_n$ we have that $H^n(U_{i_0,\dots,i_n}, \mathcal{F}) = 0$ for all $n > 0$. Then*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

for all n.

Proof. We now consider the double complex above. The assumptions ensure that the double complex satisfies the assumptions of the Lemma. Thus we conclude using the Lemma. □

This proposition is the ultimate version of the statement that one can resolve the source and the target. Before proceeding we would like to give an Example from Topology so that the reader can get a feeling for sheaf cohomology. This Example is not important for anything that follows and can safely be ignored.

Example 11.19. Let X be a nice topological space, i.e. a CW complex or a manifold. Then we would like to consider the sheaf cohomology $H^*(X, \mathbb{Z})$ where $\mathbb{Z}$ is the sheaf of locally constant $\mathbb{Z}$ valued functions on X (i.e. continuous wrt the discrete topology on $\mathbb{Z}$). It turns out that this is isomorphic to singular cohomology of X, which is an invariant that we would like to compute. In particular it follows that if X is contractible, then $H^n(X, \mathbb{Z}) = 0$ for $n > 0$ and $H^0(X, \mathbb{Z}) = \mathbb{Z}(X)$ is free on the connected components of X. Now assume that we have an ordered covering $(U_i)_{i \in I}$ of X with the property that all $U_{i_0,\dots,i_n}$ are also nice and contractible (such covers are called *good*). For example we could let X be a manifold and choose a covering by small balls (with respect to some Riemannian metric). For example we could try to do this for a circle S^1 in which case one needs three open sets and the chain complex takes the form

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0 \to 0 \to ...$$

where the differentials are given as...

COROLLARY 11.20. *Let X be a separated scheme, $\mathcal{U}$ be a cover by affine opens and $\mathcal{M}$ a quasi-coherent sheaf. Then*

$$\check{H}^n(\mathcal{U}, \mathcal{M}) \cong H^n(X, \mathcal{M})$$

for all n.

PROOF. In a separated scheme the intersection of affine opens is affine (see Proposition 6.11). The restriction of a quasi-coherent sheaf is quasi-coherent. Thus the claim follows from Proposition 11.18 combined with Theorem 11.7. $\square$

12. Finiteness of cohomology

Recall from Definition 7.1 that projective schemes over some base A are those schemes that admit a closed immersion into $\mathbb{P}^n_A$.

THEOREM 12.1. *Let X be a projective scheme over some noetherian ring and $\mathcal{F}$ be a coherent sheaf. Then for all $k \geq 0$ the cohomology group $H^k(X, \mathcal{F})$ is a finitely generated A-module and vanishes for $k \gg 0$.*

PROOF. We first want to prove the vanishing in high degrees. To this end we note that X is separated as a closed subscheme of the separated scheme $\mathbb{P}^n_A$. Moreover $\mathbb{P}^n_A$ has a finite cover by $n+1$-affine open sets. The intersections of X with the opens are still affine (as closed subsets of the affine opens) and thus we can cover X by $(n+1)$ affine open sets, which implies that the cohomology can be computed by the Čech cohomology wrt this cover. This cohomology evidently vanishes in degree $> n$ proving the second claim of the Theorem.
Now for the first part of the Theorem we need some pre considerations about closed immersion: We first observe that for a closed immersion $i : X \to Y$ of schemes the induced functor $i_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$ is exact. In fact the stalks of the pushforward $i_*(\mathcal{F})_y$ are simply given by the stalks $\mathcal{F}_y$ for $y \in X$ and 0 else. Moreover if we have an injective sheaf $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ then it is flasque, as we have seen before. But the pushforward $i_*(\mathcal{F})$ of any flasque sheaf is again flasque. Thus it follows that we can compute the sheaf cohomology of $\mathcal{F}$ as the sheaf cohomology of $i_*(\mathcal{F})$. Moreover the pushforward of a coherent sheaf is also coherent if Y is noetherian (otherwise we would need conditions on i). This can again be checked locally.
Together these considerations imply that we have that

$$H^k(X, \mathcal{F}) \cong H^k(\mathbb{P}^n_A, i_*\mathcal{F})$$

Thus we may without loss of generality assume that $X = \mathbb{P}^n_A$.
Now we would like to downward induct on k. For some large d we have that $\mathcal{F} \otimes \mathcal{O}(d)$ admits an epimorphism

$$\mathcal{O}^m \twoheadrightarrow \mathcal{F} \otimes \mathcal{O}(d)$$

by Corollary 7.21. Equivalently we have an epimorphism $\mathcal{O}(-d)^m \twoheadrightarrow \mathcal{F}$ and therefore a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(-d)^m \to \mathcal{F} \to 0$$

where $\mathcal{G}$ is coherent as well ans we look at the associated long exact sequence

$$... \to H^k(\mathbb{P}^n_A, \mathcal{O}(-d)^m) \to H^k(\mathbb{P}^n_A, \mathcal{F}) \to H^{k+1}(\mathbb{P}^n_A, \mathcal{G}) \to ...$$

By inductive assumption the right hand term is finitely generated. The first term is also finitely generated by the next computation. This then implies the claim. □

Now we would like to state the computation of the cohomology of projective space $\mathbb{P}^n$ with values in the twisted sheaves $\mathcal{O}(d)$. To this end we want to establish some terminology. We consider the graded ring $A[x_0, .., x_n]$ with x_i in degree 1. Then the d-th graded piece is given by homogenous polynomials of degree d, i.e. has an A basis consisting of monomials $x_0^{d_0} \cdot ... \cdot x_n^{d_n}$ with $\sum_{i=0}^n d_i = d$. Similarly we let $A[\frac{1}{x_0}, .., \frac{1}{x_n}]$ be the graded ring of polynomials in x_i^{-1} which accordingly have degree -1 and $\frac{1}{x_0 \cdot ... \cdot x_n} A[\frac{1}{x_0}, .., \frac{1}{x_n}]$ be an ideal in this graded ring with the induced grading. We have that the d-th graded component of this graded A-module vanishes for $d \geq -n$ and has a basis consisting of monomials $x_0^{-d_0} \cdot ... \cdot x_n^{-d_n}$ with $d_i \geq 1$ and $\sum_{i=0}^n d_i = -d$.

PROPOSITION 12.2. *For $d \in \mathbb{Z}$, $n \geq 1$ and A arbitrary we have*

$$H^k(\mathbb{P}^n_A, \mathcal{O}(d)) = \begin{cases} A[x_0, .., x_n]_d. & \text{for } k = 0 \\ \left(\frac{1}{x_0 \cdot ... \cdot x_n} A[\frac{1}{x_0}, .., \frac{1}{x_n}]\right)_d & \text{for } k = n \\ 0 & \text{else} \end{cases}$$

where each x_i has degree 1.

Note that only for $d \geq 0$ only the first term is non-zero and for $d < -n$ only the latter. In between both terms vanish. In order to prove this result we would like to quickly talk about graded modules and graded rings.

DEFINITION 12.3. *A graded ring is given by a sequence of abelian groups $(R_i)_{i \in \mathbb{Z}}$ together with multiplication maps $R_i \otimes R_j \to R_{i+j}$ such that $\bigoplus_{i \in \mathbb{Z}} R_i$ with those maps becomes a ring. A graded module is given by a sequence of abelian groups $(M_i)_{i \in \mathbb{Z}}$ with maps $M_i \otimes R_j \to M_{i+j}$ such that $\bigoplus M_i$ becomes an R-module.*

EXAMPLE 12.4. Consider the ring $\mathbb{A}[x_0, ..., x_n]$ with x_i in degree 1 and A in degree 0. The i-th graded piece consists of those polynomials of degree i.
Also the quotient

$$A[x_0, ..., x_n]/(f_1, ..., f_k)$$

where all f_i are homogenous polynomials is a graded ring with the induced grading. For any graded ring R and $n \in \mathbb{Z}$ we have the graded module

$$R(n)_i = R_{i+n}$$

given by shifting. More generally for any graded R-module M we can define

$$M(n)_i = M_{i+n} \ .$$

DEFINITION 12.5. *Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded ring.*

(1) *We set $R^+ = \bigoplus_{i > 0} R_i$*
(2) *an ideal $I \subseteq R$ is called homogenous, if it is generated by homogenous elements. Equivalently, if $I = \bigoplus_{i \in \mathbb{Z}} I \cap R_i$.*
(3) *The set* $\operatorname{Proj}(R)$ *is defined as the set of homogenous prime ideals $\mathfrak{p} \subseteq R$ with $R^+ \not\subseteq \mathfrak{p}$.*

(4) *For any homogenous ideal $I \subseteq R$ we define the subset $V_+(I) \subseteq \mathrm{Proj}(R)$ as the set of those prime ideals that contain I.*
(5) *For a homogenous element $f \in R$ of positive degree (i.e. $f \in R_i$ for $i > 0$) we define*
$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid f \notin \mathfrak{p}\}$$
(6) *For a graded module M and a homogenous element f of positive degree we set*
$$\widetilde{M}(D_+(f)) := M[f^{-1}]_0 \ .$$
(7) *We set $\mathcal{O}_{\mathrm{Proj}(R)} = \widetilde{R}(D_+(f)) = R[f^{-1}]_0$.*

Proposition 12.6. (1) $\mathrm{Proj}(R)$ *is a topological space with closed sets $V_+(I)$. It is a subspace of* $\mathrm{Spec}(R)$ *and has a basis of the topology consisting of the $D_+(f)$,*
(2) *The constructions $D_+(f) \mapsto \widetilde{M}(D_+(f))$ canonically extends to sheaf of abelian groups on* $\mathrm{Proj}(R)$
(3) *The pair $(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj(R)}})$ is a scheme, which is separated.*
(4) *The standard opens $D_+(f)$ are affine and thus isomorphic to* $\mathrm{Spec}(R[f^{-1}]_0)$.
(5) *$\widetilde{M}$ is a quasi-coherent sheaf*
(6) *There is a canonical morphism* $\mathrm{Proj}(R) \to \mathrm{Spec}(R_0)$ *of schemes induced by taking a homogenous ideal to the intersection with R_0.*

Proof. Omitted. □

Warning 12.7. *In general* $\mathrm{Proj}(R)$ *is not quasi-compact, e.g.* $\mathrm{Proj}(\mathbb{Z}[x_1, x_2, ...])$ *with x_i in degree 1. But it is quasi-compact if R is generated by finitely many elements in degree* 1.
Also in general a quasi-coherent sheaf on $\mathrm{Proj}(R)$ *is not necessarily of the form $\widetilde{M}$. But if* $\mathrm{Proj}(R)$ *is quasi-compact this is the case, see Stacks 28.28.5. If moreover* $\mathrm{Proj}(R)$ *is covered by standard opens $D_+(f)$ for degree* 1 *elements f then we also have that $\Gamma(X, \widetilde{M(n)}) = M_n$, which gives an important formula for global sections. We also do not have that $\widetilde{M \otimes_R N} = \widetilde{M} \otimes_{\mathcal{O}_{\mathrm{Proj(R)}}} \widetilde{N}$, see Remark 27.9.2.*

Example 12.8. For the ring $R = A[X_0, ..., X_n]$ the scheme Proj(R) is given by projective space $\mathbb{P}^n_A$ with $D_+(x_i) = U_i$ the standard opens given by
$$\mathrm{Spec}(A[x_0, ..., x_n][x_i^{-1}]_0) = \mathrm{Spec}(A[x_0/x_i, ..., \widehat{x}_i/x_i, ..., x_n/x_i]) \cong \mathbb{A}^n_A \ .$$
We have that $\widetilde{R(n)} = \mathcal{O}(n)$. Generally we have that $U_{i_0,...,i_k} = D_+(x_{i_0} \cdot ... \cdot x_{i_k})$ and thus
$$\mathcal{O}(d)(U_{i_0,...,i_k}) = A\Big[X_0, ..., X_n, \frac{1}{X_{i_0} \cdot ... \cdot X_{i_k}}\Big]_d$$
If we have a number of homogenous $f_1, .., f_k \in A[X_0, ..., X_n]$ then we have that
$$\mathrm{Proj}(A[X_0, ..., X_n]/f_1, .., f_k) \cong V(f_1, ..., f_k) \subseteq \mathbb{P}^n_A \ .$$

Proof of Proposition 12.2. We use the Cech complex for the standard cover $\mathbb{P}^n_A = \bigcup_{i=0}^n U_i$ by affines to compute the cohomology with the standard ordering on the index set $\{0, ..., n\}$. The Čech complex takes the form
$$\check{C}^k(\mathcal{U}, \mathcal{O}(d)) = \bigoplus_{i_0 < i_1 < ... < i_k} A\Big[X_0, ..., X_n, \frac{1}{X_{i_0} \cdot ... \cdot X_{i_k}}\Big]_d$$

with differential given by the alternating sum of canonical maps to the localizations:

$$d(s)_{i_0,\ldots,i_k} = \sum_{j=0}^{k} (-1)^k s_{i_0,\ldots,\hat{i}_j,\ldots,i_n}$$

We can further consider this complex as being graded over the degree of monomoals $\overrightarrow{e} = (e_0, ..., e_n) \in \mathbb{Z}^n$ with $\sum_{i=0}^n e_i = d$. In other words we declare that the monomoial $X_0^{e_0} \cdot ... \cdot X_n^{e_n}$ has degree $\overrightarrow{e} = (e_0, ..., e_n)$. We shall also write this monomial as $X^{\overrightarrow{e}}$. Clearly the differential of the complex respects this degree and the whole complex decomposes into a direct sum over subcomplexes

$$\check{C}(\mathcal{U}, \overrightarrow{e}) \subseteq \check{C}(\mathcal{U}, \mathcal{O}(d)) \ .$$

Thus we have that

$$H^*(\mathbb{P}^n_A, \mathcal{O}(d)) = H^*(\check{C}^k(\mathcal{U}, \mathcal{O}(d))) = \bigoplus_{\overrightarrow{e} \text{ with } \sum_{i=0}^n e_i = d} H^*(\check{C}(\mathcal{U}, \overrightarrow{e}))$$

Now we distinguish three cases:

(1) All $e_i < 0$. This can of course only happen if $d \leq -n$. In this case we find

$$\check{C}(\mathcal{U}, \overrightarrow{e}) = \big(0 \to 0 \to ... \to 0 \to A \cdot X^{\overrightarrow{e}} \to 0 \to ...\big)$$

where the only non-trivial entry sits in degree n and is isomorphic to A. Thus we find that

$$H^*(\check{C}(\mathcal{U}, \overrightarrow{e})) = \begin{cases} \Big(\frac{1}{x_0 \cdot \ldots \cdot x_n} A[\frac{1}{x_0}, .., \frac{1}{x_n}]\Big)_e & \text{for } * = n \\ 0 & \text{else} \end{cases}$$

(2) All $e_i \geq 0$. This can only happen if $d \geq 0$. Now the complex takes the form

$$\check{C}(\mathcal{U}, \overrightarrow{e})) = \Big(\bigoplus_{i_0} A \cdot X^{\overrightarrow{e}} \to \bigoplus_{i_0 < i_1} A \cdot X^{\overrightarrow{e}} \to \bigoplus_{i_0 < i_1 < i_2} A \cdot X^{\overrightarrow{e}} \to \ldots\Big)$$

This complex is isomorphic to the complex

$$\bigoplus_{i_0} A \to \bigoplus_{i_0 < i_1} A \to \bigoplus_{i_0 < i_1 < i_2} A \to \ldots$$

which arises as the Čech complex of $\mathcal{O}_{\mathrm{Spec}(A)}$ for the cover consisting of the $(n+1)$ open sets V_i all given by $\mathrm{Spec}(A)$. Thus we find that it only has cohomology in degree 0. More precisely

$$H^*(\check{C}(\mathcal{U}, \overrightarrow{e})) = \begin{cases} A[x_0, .., x_n]_e & \text{for } * = 0 \\ 0 & \text{else} \end{cases}$$

(3) Finally if precisely $n-k$ of the numbers $e_0, ..., e_n$ are negative. For simplicity we assume these are $e_{k+1}, ..., e_n$, the other cases are a relabelling. Then the complex only contains those summands corresponding to the entries $i_0 < i_1 < ... < i_s$ that end with $k+1 < ... < n$. Explicitly it takes the form

$$\check{C}(\mathcal{U}, \overrightarrow{e}) = \big(0 \to ... \to 0 \to AX^{\overrightarrow{e}} \to \bigoplus_{i_0 < k+1} AX^{\overrightarrow{e}} \to \bigoplus_{i_0 < i_1 < k+1} AX^{\overrightarrow{e}} \to ...\big)$$

where the first entry is in degree $n-k-1$. This complex is up to shifting isomorphic to the complex

$$(A \to \bigoplus_{i_0<k+1} A \to \bigoplus_{i_0<i_1<k+1} A \to \bigoplus_{i_0<i_1<i_2<k+1} A \to \dots$$

If it wasn't for the first A it would be the Čech complex of $\mathcal{O}_{\mathrm{Spec}(A)}$ for the cover consisting of the $(k+1)$ open sets V_i all given by $\mathrm{Spec}(A)$ (similar to the second case). The further A in degree zero makes our complex the augmented Čech complex and thus has vanishing cohomology, i.e.

$$H^*(\check{C}(\mathcal{U}, \overrightarrow{e})) = 0$$

Finally the cases (1)-(3) show the result. □

We would like to finish by quoting a result about the vanishing of sheaf cohomology.

THEOREM 12.9. *If X is a spectral space*[6] *e.g. the topological space underlying a qcqs scheme, and $\mathcal{F}$ a sheaf of abelian groups, then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

PROOF. Proposition 0A3G □

13. Riemann–Roch

Recall the notion of an effective Cartier divisor, which is given by a closed subscheme $Z \subseteq C$ such that the associated ideal sheaf $\mathcal{I}_Z \in \mathrm{QCoh}(C)$ is a line bundle. Then we denote the dual by $\mathcal{O}(Z)$.

LEMMA 13.1. *Let C be a smooth curve over some field k, i.e. a smooth variety of Krull dimension 1. Every closed point $x \in C$ defines (with the reduced subscheme structure) an effective Cartier divisor.*

PROOF. We may assume that $C = \mathrm{Spec}\, R$, since everything is local (and away from the point the vanishing ideal is trivial). Let I be the vanishing ideal of x. We have seen in the Exercises that

$$I/I^2 = \Omega^1_{C/k} \otimes_R \kappa(x)$$

Thus it follows from smootheness that I/I^2 is 1-dimensional as a $\kappa(x)$ vector space. Let f be an element of I that maps to a generator of I/I^2. By Nakayama this is even a generator of I in a neighborhood of x, that is after replacing R by some localization we have that I is a principal ideal. Since R is a domain it follows that I is a line bundle. □

Note that this crucially needs the smoothness of C, otherwise it would be totally false. From Proposition 8.11 we immediately deduce:

COROLLARY 13.2. *Let C be a smooth curve over some field k. Then every line bundle on X is isomorphic to a tensor product of line bundles $\mathcal{O}(x_1)^{\otimes k_1} \otimes \dots \otimes \mathcal{O}(x_n)^{\otimes k_n}$ for closed points $x_i \in C$ and integers $k_i \in \mathbb{Z}$.*

[6]Recall that a spectra space is a topological space that is sober, quasi-compact, quasi-separated and the quasi-compact opens form a basis of the topology

From now on we let C be a smooth and projective curve over an algebraically closed field k. By this we mean a 1-dimensional, projective variety which is smooth. If we speak about points $x \in C$ they are assumed to be closed for this section (unless we specifiy it to be the generic point). Note that by the results of the previous section we have that all cohomology groups $H^k(C, \mathcal{O}_C)$ are zero unless $k = 0, 1$ in which case these are finite dimensional vector spaces.
We claim

$$H^0(C, \mathcal{O}_C) = \mathcal{O}_C(C) = k$$

i.e. all global sections are constant. This follows from the more general assertion:

PROPOSITION 13.3. *Let X be a projective variety over k, i.e. a closed subvariety of $\mathbb{P}^n_k$. Then $\mathcal{O}_X(X) = k$.*

PROOF. $H^0(C; \mathcal{O}_X)$ is a finitely-generated k-module by Theorem 12.1. It is also a domain: If $f \cdot g = 0$, then $X = V(f) \cup V(g)$, so by irreducibility either of those is all of X, say $V(f)$. Then f is nilpotent, hence zero since X is reduced. It follows that $H^0(C; \mathcal{O}_X)$ is a finite dimensional k-algebra without zero divisors, hence a finite field extension. By algebraic closedness, $H^0(C; \mathcal{O}_X) = k$. □

Note that this claim would be false if we drop the condition that X is irreducible, then we could for example have two points.

DEFINITION 13.4. *We define the genus of C to be*

$$g := \dim_k H^1(C, \mathcal{O}_C) \ .$$

Informally the genus measures the number of 'holes'. For example if $k = \mathbb{C}$ then curves over $\mathbb{C}$ are orientable surfaces over $\mathbb{R}$ and those have a number of holes, which is the genus. Note that the finite dimensionality of the first cohomology is crucial for this definition. Generally the finite dimensionality is key for everything in this section.

EXAMPLE 13.5. The genus of $\mathbb{P}^1$ is 0 since $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

DEFINITION 13.6. *A (Weil) divisor on C is a formal sum*

$$D = \sum_{x \in C} n_x[x]$$

where almost all n_x are zero. The group $\mathrm{Div}(C)$ *is the group of divisors under addition. Equivalently it is the free abelian group generated by the closed points of C. A divisor is called* effective *if all n_x are non-negative.*
For a divisor D we define the sheaf $\mathcal{O}(D)$ as the tensor product

$$\mathcal{O}(D) = \bigotimes_{c \in C} \mathcal{O}(x)^{n_x} \ .$$

The association $D \mapsto \mathcal{O}(D)$ induces a group homomorphism

$$\mathrm{Div}(C) \to \mathrm{Pic}(C)$$

which is surjective by Corollary 13.2.

REMARK 13.7. Every effective Weil divisor gives rise to an effective Cartier divisor with associated sheaf $\mathcal{O}(D)$ and sections given by the tensor product of sections (which exist for positive integers). This follows from the fact that effective Cartier divisors are essentially the same as line bundles with a section. The subspace of the associated Cartier divisor is the union of all the points, but the subscheme structure takes care of multiplicities.

Our goal eventually is to give a more direct definition of $\mathcal{O}(D)$ for a divisor D and the relation to meromorphic sections. For the moment we shall stick with the current definition.

Definition 13.8. *The degree of a divisor*

$$D = \sum_{x \in C} n_x[x]$$

is given by

$$\deg(D) = \sum_{x \in C} x_n$$

This defines a homomorphism

$$\deg : \mathrm{Div}(C) \to \mathbb{Z}\ .$$

For a line bundle $\mathcal{L}$ we define the degree as the degree of a divisor D with $\mathcal{O}(D) = \mathcal{L}$.

Theorem 13.9 (Riemann–Roch I). *The degree of a line bundle is well-defined and for any line bundle $\mathcal{L}$ on C we have*

$$\dim_k H^0(C, \mathcal{L}) - \dim_k H^1(C, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

Proof. We first assume that $\mathcal{L} = \mathcal{O}(D)$ for $D = \sum_{x \in C} n_x[x]$ and prove the Riemann-Roch formula.

$$\dim_k H^0(C, \mathcal{O}(D)) - \dim_k H^1(C, \mathcal{O}(D)) = \sum_{x \in C} n_x + 1 - g. \tag{17}$$

If $D = 0$ then we get that $\mathcal{O}(D) = \mathcal{O}_C$ and so we get that $\dim_k H^0(C, \mathcal{O}) = 1$ and $\dim_k H^1(C, \mathcal{O}) = g$ so that the equation (17) is true.
For the next step assume that $D = D' + x$. Then we have

$$\mathcal{O}(D) = \mathcal{O}(D') \otimes \mathcal{O}(x)$$

For $\mathcal{O}(x)$ we have a short exact sequence

$$\mathcal{O} \to \mathcal{O}(x) \to i_*(k)$$

where i is the inclusion $\{x\} \to C$ of the point (with the reduced scheme structure). This follows from the short exact sequence

$$\mathcal{I} \to \mathcal{O} \to i_*(k)$$

(given affin locally by $I \to R \to R/I = k$) by tensoring with $\mathcal{I}^{-1} = \mathcal{O}(x)$. Tensoring further with $\mathcal{O}(D')$ we get a short exact sequence

$$\mathcal{O}(D') \to \mathcal{O}(D) \to i_*(k) \otimes \mathcal{O}(D') \cong i_*(k)$$

Thus we deduce that we get a long exact sequence

$$0 \to H^0(C, \mathcal{O}(D')) \to H^0(C, \mathcal{O}(D)) \to k \to H^1(C, \mathcal{O}(D')) \to H^1(C, \mathcal{O}(D)) \to 0$$

From this we see that we have short exact sequences

$$0 \to H^0(C, \mathcal{O}(D')) \to H^0(C, \mathcal{O}(D)) \to \mathrm{im} \to 0$$

and

$$0 \to k/\mathrm{im} \to H^1(C, \mathcal{O}(D')) \to H^1(C, \mathcal{O}(D)) \to 0$$

so that

$$\dim H^0(C, \mathcal{O}(D')) + \dim_k \mathrm{im} = \dim H^0(C, \mathcal{O}(D))$$
$$1 - \dim \mathrm{im} + \dim H^1(C, \mathcal{O}(D)) = \dim H^1(C, \mathcal{O}(D'))$$

so that we get

$$\begin{aligned}
&\dim H^0(C, \mathcal{O}(D)) - \dim H^1(C, \mathcal{O}(D)) \\
&= \dim H^0(C, \mathcal{O}(D')) + \dim \mathrm{im} - \dim H^1(C, \mathcal{O}(D')) - \dim \mathrm{im} + 1 \\
&= \dim H^0(C, \mathcal{O}(D')) - \dim H^1(C, \mathcal{O}(D')) + 1
\end{aligned}$$

Now assume formula (17) holds for either D or D', then we get from this equality that it also holds for the other, since the right hand sides of (17) differ also exactly by 1.

Altogether we see inductively that formula (17) holds for all Divisors D. Now every line bundle $\mathcal{L}$ is of the form $\mathcal{O}(D)$ and we can define

$$\deg(\mathcal{L}) := \dim_k H^0(C, \mathcal{L}) - \dim_k H^1(C, \mathcal{L}) - 1 + g$$

This is obviously an invariant of $\mathcal{L}$ and by the first part extends the degree of divisors. This finishes the proof. □

The term $H^1(C, \mathcal{L})$ can be reinterpreted using the following theorem, which we state without proof:

Theorem 13.10 (Serre duality). *Let $\mathcal{V}$ be a vector bundle over C. Then for $i = 0, 1$ there are canonical isomorphisms*

$$H^i(C, \Omega^1_{C/k} \otimes \mathcal{V}^\vee) \cong H^{1-i}(X, \mathcal{V})^\vee \ .$$

in particular we get

$$\dim_k H^1(C, \mathcal{V}) = \dim_k H^0(C, \Omega^1_{C/k} \otimes \mathcal{V}^\vee) \ .$$

Using this result we can rewrite the Riemann-Roch theorem without sheaf cohomology:

Theorem 13.11 (Riemann–Roch II). *We have*

$$\dim_k \Gamma(C, \mathcal{L}) - \dim_k \Gamma(C, \mathcal{L}^\vee \otimes \Omega^1_{C/k}) = \deg(\mathcal{L}) + 1 - g.$$

Example 13.12. For the line bundle $\mathcal{L} = \Omega^1_{C/k}$ we get that

$$\dim_k \Gamma(C, \Omega^1_{C/k}) - \dim_k \Gamma(C, \mathcal{O}) = \deg(\Omega^1_{C/k}) + 1 - g$$

We can calculate all terms, the first one is by Serre duality g so that the lhs is $g - 1$. Thus we get that

$$\deg(\Omega^1_{C/k}) = 2g - 2 \ .$$

Note that it is not obvious how to write $\Omega^1_{C/k}$ as $\mathcal{O}(D)$!

We record the following important characterisation of degree, which was implicit in the proof of Riemann-Roch:

Lemma 13.13. *Let C be a smooth projective curve as above, and let $s : \mathcal{L} \to \mathcal{L}'$ be a nonzero morphism of line bundles on C. Then*

$$\deg(\mathcal{L}') - \deg(\mathcal{L}) = \dim \mathcal{O}_{V(s)}(V(s)).$$

PROOF. By definition of $V(s)$, we have an exact sequence

$$0 \to \mathcal{L} \otimes (\mathcal{L}')^{-1} \to \mathcal{O}_C \to i_*\mathcal{O}_{V(s)} \to 0,$$

Tensoring with $\mathcal{L}'$ and using the projection formula, this translates to

$$0 \to \mathcal{L} \to \mathcal{L}' \to i_*i^*\mathcal{L}' \to 0,$$

Since $V(s)$ is a closed subset of C, and C is irreducible of dimension 1, $V(s)$ is zero-dimensional, and thus consists of finitely many points $x_1, \ldots, x_n$. Around each point, the line bundle $i^*\mathcal{L}'$ is trivial, so we get an isomorphism $\mathcal{O}_{V(s)} \to i^*\mathcal{L}'$. Hence we have:

$$H^0(C; i_*i^*\mathcal{L}') = H^0(V(s); i^*\mathcal{L}') \cong H^0(V(s); \mathcal{L}'_{V(s)}) = H^0(V(s), \mathcal{O}_{V(s)}),$$

and $H^1(C; i_*i^*\mathcal{L}') \cong H^1(V(s); i^*\mathcal{L}') = 0$ since $V(s)$ is 0-dimensional. So we have a long exact sequence

$$0 \to H^0(C;\mathcal{L}) \to H^0(C;\mathcal{L}') \to H^0(V(s), \mathcal{O}_{V(s)}) \to H^1(C;\mathcal{L}) \to H^1(C;\mathcal{L}') \to 0.$$

As in the proof of Riemann-Roch, this implies that the alternating sum of dimensions is 0, which we can rearrange as

$$(\dim H^0(C;\mathcal{L}) - \dim H^1(C;\mathcal{L})) - (\dim H^0(C;\mathcal{L}') - \dim H^1(C;\mathcal{L}')) + \dim H^0(V(s), \mathcal{O}_{V(s)} = 0.$$

Combining this with Riemann-Roch, we obtain

$$\deg(\mathcal{L}) - \deg(\mathcal{L}') = -\dim H^0(V(s), \mathcal{O}_{V(s)}),$$

i.e. the desired result. □

COROLLARY 13.14. *If there is a nonzero morphism $s : \mathcal{L} \to \mathcal{L}'$, then $\deg(\mathcal{L}') \geq \deg(\mathcal{L})$, with equality if and only if s is invertible. In particular:*

(1) *Line bundles of negative degree have no nonzero sections, line bundles of degree 0 have nonzero sections only if they are trivial.*
(2) *The relation defined by $\mathcal{L}' \geq \mathcal{L}$ if there exists a non-trivial morphism $\mathcal{L} \to \mathcal{L}'$ is a partial order on isomorphism classes of line bundles.*

PROOF. From $\deg(\mathcal{L}') - \deg(\mathcal{L}) = \dim \mathcal{O}_{V(s)}(V(s))$, the inequality is clear. If we have equality, i.e. $\mathcal{O}_{V(s)}(V(s)) = 0$, we need to have $V(s)$ empty, since it is a union of finitely many points, hence affine. Then s is invertible, as claimed. □

COROLLARY 13.15. *For a line bundle $\mathcal{L}$ we have that*

$$\dim_k \Gamma(C, \mathcal{L}) \geq \deg(\mathcal{L}) + 1 - g$$

with equality if $\deg(\mathcal{L}) > 2g - 2$.

PROOF. The inequality clearly follows from Riemann–Roch. For equality we have to show that

$$\Gamma(\mathcal{L}^\vee \otimes \Omega^1_{C/k}) = 0$$

If $\deg(\mathcal{L}) > 2g - 2$ then we have that

$$\deg(\mathcal{L}^\vee \otimes \Omega^1_{C/k}) = 2g - 2 - \deg(\mathcal{L}) < 0$$

so that this cannot admit a section. □

COROLLARY 13.16. *A line bundle $\mathcal{L}$ of degree $\deg(\mathcal{L}) \geq 2g$ is globally generated, i.e. there exists an epimorphism $\bigoplus \mathcal{O} \to \mathcal{L}$.*

Proof. It suffices to check that for any point $x \in C$, there exists a section $\mathcal{O} \to \mathcal{L}$ which does not vanish at x, since then this section is automatically an isomorphism in a neighbourhood of x, and by quasicompactness finitely many such sections suffice to generate $\mathcal{L}$.
Letting $i : \operatorname{Spec} k \to C$ denote the inclusion of x, we have an exact sequence

$$0 \to \mathcal{O}(-x) \to \mathcal{O} \to i_*k \to 0,$$

which we may tensor with $\mathcal{L}$ to obtain a sequence

$$0 \to \mathcal{L} \otimes \mathcal{O}(-x) \to \mathcal{L} \to i_*i^*\mathcal{L} \to 0.$$

The resulting morphism $H^0(C; \mathcal{L}) \to H^0(C; i_*i^*\mathcal{L}) \cong H^0(\operatorname{Spec} k; i^*\mathcal{L}) = \mathcal{L}_x \otimes_{\mathcal{O}_x} \kappa(x)$ is evaluation at x. By assumption, $\deg(\mathcal{L} \otimes \mathcal{O}(-x)) > 2g-2$, so $H^1(C; \mathcal{L} \otimes \mathcal{O}(-x)) = 0$, and thus the evaluation map is surjective. We thus obtain a section of $\mathcal{L}$ not vanishing at x, as desired. $\square$

Corollary 13.17. *A line bundle $\mathcal{L}$ is ample iff $\deg(\mathcal{L}) > 0$.*

Proof. We need to show that the $D(s)$ for $s \in \mathcal{L}^{\otimes k}(C)$ form a basis for the topology. In other words, we need to show that for any $x \in C$ and any open neighbourhood $x \in U \subseteq C$, we find $s \in \mathcal{L}^{\otimes k}(C)$ with $x \in D(s) \subseteq U$. The complement of U is a closed subset of C, and since C is irreducible of dimension 1, it consists of finitely many points $x_1, \ldots, x_n$. We are thus looking for s which does not vanish at x, but does vanish at $x_1, \ldots, x_n$. If we had a section $f : \mathcal{O} \to \mathcal{L}^{\otimes k} \otimes \mathcal{O}(-x_1 - \ldots - x_n)$ which does not vanish at x, then we could precompose with the morphism $\mathcal{L}^{\otimes k} \otimes \mathcal{O}(-x_1 - \ldots - x_n) \to \mathcal{L}^{\otimes k}$ which does vanish at $x_1, \ldots, x_n$, but not at x, obtaining a section as desired.
Choosing k large enough, such that $k \cdot \deg(\mathcal{L}) - n > 2g - 1$, we find a section f as desired. $\square$

14. Meromorphic functions and applications

Let C be a smooth curve over an algebraically closed field k (really for the first part we only need a reduced and irreducible scheme, but we will gradually need more). Recall from Proposition 6.20 that all the fraction fields of $\mathcal{O}_C(U)$ for affine opens $U \subseteq C$ are isomorphic and given by $\kappa(\eta)$ for the generic point η.

Definition 14.1. *We call the field $K = K(C) = \kappa(\eta)$ for the generic point $\eta \in C$ the function field of C or the field of meromorphic functions.*

More generally, we want to think of an element of $\mathcal{L}_\eta$ for a line bundle $\mathcal{L}$ as a "meromorphic section of $\mathcal{L}$". Explicitly such an element is represented by a pair (U, f) with $U \subseteq C$ open, $f \in \mathcal{L}(U)$, and $(U, f) \sim (V, g)$ if f and g agree on any open subset of $U \cap V$. We justify calling these meromorphic sections through the following lemma:

Lemma 14.2. *For any open $U \subseteq C$, the map $\mathcal{L}(U) \to \mathcal{L}_\eta$ is injective, where $\eta \in C$ is the generic point. For any $f \in \mathcal{L}_\eta$, there exists a unique maximal $U \subseteq C$ with $f \in \mathcal{L}(U)$.*

Proof. If $f \in \mathcal{L}(U)$ maps to zero in $\mathcal{L}_\eta$, then it vanishes at η. But then $V(f)$ contains the closure $\overline{\eta}$, so $V(f) = C$ and f is nilpotent, hence zero.
For the second claim, if $f_i \in \mathcal{L}(U_i)$ map to the same element, then by the first claim, they agree on all intersections $U_i \cap U_j$, hence glue to an element of $\mathcal{L}(\bigcup U_i)$. $\square$

We call U the *domain of definition* of f. Since a nonempty open $U \subseteq C$ is the complement of finitely many points $x_1, \ldots, x_n$ (see the proof of Proposition 8.11 for an argument), we think of f as having poles at $x_1, \ldots, x_n$, and will call $\{x_1, \ldots, x_n\}$ the set of poles of f.
Note that although every nonzero morphism $\mathcal{L} \to \mathcal{L}'$ induces an isomorphism on $\mathcal{L}_\eta \to (\mathcal{L}')_\eta$ where x is the generic point, hence we may identify meromorphic sections of $\mathcal{L}$ with meromorphic sections on $\mathcal{L}'$, the domain of definition of a given element depends on the line bundle we look at. For example, consider $\mathcal{O}(-x) \to \mathcal{O}$. The section $1 \in \mathcal{O}(C)$ defines an element of $\mathcal{O}_\eta \cong \mathcal{O}(-x)_\eta$. As section of $\mathcal{O}$, its domain of definition is all of C. But since 1 does not lift to a global section of $\mathcal{O}(-x)$ (having no global sections since its degree is -1), its domain of definition as section of $\mathcal{O}(-x)$ is only $C \setminus \{x\}$, as it can't be all of C, but definitely contains $C \setminus \{x\}$ since $\mathcal{O}(-x) \to \mathcal{O}$ is invertible there.

DEFINITION 14.3. *Let $f \in \mathcal{L}_\eta$ be a meromorphic section. For a closed point $x \in C$, we define the order of f at $x \in C$ as the largest n for which the domain of definition of f regarded as meromorphic section of $\mathcal{L} \otimes \mathcal{O}(-n[x])$ contains x. We write $v_x(f; \mathcal{L})$ for the order of f at x (or $v_x(f)$ if the line bundle is clear).*

Assume $f \in \mathcal{L}_\eta$ is a meromorphic section and $x \in C$ a closed point. Let U be a neighbourhood of x such that the domain of definition of f contains $U \setminus \{x\}$. Recall that

$$\mathcal{L}(U \setminus \{x\}) = \operatorname{colim}_k(\mathcal{L} \otimes \mathcal{O}(k[x]))(U).$$

Here all of the maps are injective, so the colimit is really the union. The order of f at x is the largest n such that f is contained in $(\mathcal{L} \otimes \mathcal{O}(-n[x]))(U)$.

LEMMA 14.4.
1. *For $f \in \mathcal{L}_\eta$, we have $v_x(f) < 0$ if and only if f has a pole at x, $v_x(f) = 0$ if and only if f is invertible at x, and $v_x(f) > 0$ if and only if f vanishes at x.*
2. *The order is multiplicative, that is*
$$v_x(f; \mathcal{L}) + v_x(g; \mathcal{L}') = v_x(fg; \mathcal{L} \otimes \mathcal{L}')$$
3. *We have*
$$v_x(f + g; \mathcal{L}) \geq \min(v_x(f; \mathcal{L}), v_x(g; \mathcal{L}))$$
with equality if the two orders are different.
4. *Assume that $f \neq 0$. Then we have that $v_x(f; \mathcal{L})$ is nonzero only at finitely many closed points, and*
$$\sum_{x \in C} v_x(f; \mathcal{L}) = \deg(\mathcal{L}).$$

PROOF. By definition, $v_x(f) < 0$ is equivalent to x not being contained in the domain of definition of f regarded as meromorphic section of $\mathcal{L}$. For the other statements let U be a neighbourhood of x in which f is defined. We observe that the long exact sequence associated to

$$0 \to \mathcal{L} \otimes \mathcal{O}(-x) \to \mathcal{L} \to i_* i^* \mathcal{L} \to 0$$

gives an exact sequence

$$H^0(U; \mathcal{L} \otimes \mathcal{O}(-x)) \to H^0(U; \mathcal{L}) \to \mathcal{L}_x \otimes \mathcal{O}_x \kappa(x).$$

Thus, f pulls back to $\mathcal{L} \otimes \mathcal{O}(-x)(U)$ if and only if its value at x vanishes.

If f and g have order n and m, then f regarded as section of $\mathcal{L} \otimes \mathcal{O}(-n[x])$ is defined and invertible at x, g regarded as section of $\mathcal{L}' \otimes \mathcal{O}(-m[x])$ is defined and invertible at x, and so $f \otimes g$ regarded as section of $\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{O}(-(n+m)[x])$ is defined and invertible at x. Hence the order of fg regarded as section of $\mathcal{L} \otimes \mathcal{L}'$ is $n+m$ at x.
For the third claim, assume $v_x(f;\mathcal{L}) \leq v_x(g;\mathcal{L})$. By multiplicativity, the claim is equivalent to $v_x(1+g/f;\mathcal{O}) \geq \min(0, v_x(g/f;\mathcal{O}))$. But this is clear: By assumption $v_x(g/f;\mathcal{O}) \geq 0$, so it is defined at x, and hence $1+g/f$ is also defined at x. Furthermore, if $v_x(g/f;\mathcal{O}) > 0$, then it vanishes at $1+g/f$, and so the value of $1+g/f$ at x is nonzero.
For the final claim, if f is a nonzero meromorphic section of $\mathcal{L}$, it has finitely many poles. Also, f^{-1} as section of $\mathcal{L}^{-1}$ has finitely many poles, so altogether the order of f at only finitely many points is nonzero. Now note that the canonical section $s_x : \mathcal{O} \to \mathcal{O}(x)$ has order 1 at x and 0 at all other points. So $f \cdot \prod s_x^{-v_x(f;\mathcal{L})}$ is a meromorphic section of $\mathcal{L} \otimes \mathcal{O}(-D)$ with $D = \sum v_x(f;\mathcal{L}) \cdot [x]$, whose order at each point is 0. So this section has domain of definition C, and is invertible at each point, hence invertible. This implies that $\mathcal{L} \otimes \mathcal{O}(-D) = \mathcal{O}$, and hence

$$\deg(\mathcal{L}) = \deg(\mathcal{O}(D)) = \sum_x v_x(f;\mathcal{L}).$$

□

Remark 14.5. Note that the basic properties of the order only used that C is a smooth variety of dimension 1. Only the invariance of $\sum_x v_x(f;\mathcal{L})$ used that C is a projective variety, through finiteness of cohomology. If C is not a projective variety, then $\sum_x v_x(f;\mathcal{L})$ can take different values for different sections of the same line bundle $\mathcal{L}$, for example $\mathcal{O}$ on $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ admits the sections 1 (with $v_x(1;\mathcal{O}) = 0$ for each x) and t with $v_x(t;\mathcal{O})$ given by 1 for the point $x = (t)$ and 0 everywhere else.

Example 14.6. By Example 13.12, we have $\deg(\Omega^1_{C/k}) = 2g-2$, meaning that for an arbitrary nonzero meromorphic section of $\Omega^1_{C/k}$, the total order is $2g-2$.
For example, write $C = \mathbb{P}^1_k$ as union of $U = \mathrm{Spec}(k[t])$ and $V = \mathrm{Spec}(k[t'])$, glued along $k[(t')^{\pm 1}] \cong k[t^{\pm 1}]$, $t' \mapsto t^{-1}$. The element $dt \in \Omega^1_{k[t]/k} = \Omega^1_{C/k}(U)$ defines a meromorphic section whose order at every point in U is zero, since dt freely generates $\Omega^1_{k[t]/k}$ as a $k[t]$-module. So the order at the remaining point of $\mathbb{P}^1_k$ has to be -2, and indeed, our section is given on V by:

$$d((t')^{-1}) = -\frac{1}{(t')^2} dt'$$

Example 14.7. For $C = V(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$, we computed in a previous exercise that $H^1(C;\mathcal{O})$ is 1-dimensional, so $g = 1$. It follows that $\deg(\Omega^1_{C/k}) = 0$. In fact, Serre duality tells us that $H^0(\Omega^1_{C/k}) \cong (H^1(\mathcal{O};k))^\vee$ is 1-dimensional, so there exists a nonzero global section of $\Omega^1_{C/k}$. Since its domain of definition is all of C, the order must be nonnegative everywhere, but since $\deg \Omega^1_{C/k} = 0$, the order must be 0 everywhere. So in fact $\Omega^1_{C/k} \cong \mathcal{O}$ here! It is not obvious how to find such an isomorphism, i.e. a global section of $\Omega^1_{C/k}$ that vanishes nowhere. In our concrete example, one can compute

$$\frac{x_2^2}{x_1^2} d\left(\frac{x_0}{x_2}\right) = -\frac{x_2^2}{x_0^2} d\left(\frac{x_1}{x_2}\right) = \frac{x_1^2}{x_0^2} d\left(\frac{x_2}{x_1}\right),$$

with domain of definition $D(x_1x_2) \cup D(x_0x_2) \cup D(x_0x_1) = C$.

COROLLARY 14.8. *Every non-zero meromorphic function $f \in K = K(C)$ has as many zeros as poles (counted with multiplicity), that is*

$$\sum_{x \in C} v_x(f, \mathcal{O}) = 0$$

PROOF. Follow from Lemma 14.4(4) since $\deg(\mathcal{O}) = 0$. □

Having developed the notion of orders, we can finally give a more intuitive description of the line bundles $\mathcal{O}(D)$.

PROPOSITION 14.9. *Let $D = \sum n_i x_i$ be a divisor. The sheaf $\mathcal{O}(D)$ admits the following explicit description:*

$$\mathcal{O}(D)(U) = \{f \in K = \kappa(\eta) = \mathcal{O}_\eta \mid v_{x_i}(f; \mathcal{O}) \geq -n_i \text{ for each } x_i \in U\}$$

PROOF. Recall that $\mathcal{O}(x)$ comes with a canonical global section s_x with $v_x(s_x; \mathcal{O}(x)) = 1$, and $v_y(s_x; \mathcal{O}(x)) = 0$ at all other points. Inverting this, $\mathcal{O}(-x)$ has a canonical meromorphic section with order -1 at x, and tensoring these together, $\mathcal{O}(D)$ has a canonical meromorphic section s_D of order n_i at x_i, where $D = \sum n_i[x_i]$.
We have an injective map

$$\mathcal{O}(D)(U) \to \mathcal{O}(D)_\eta \xrightarrow{s_D^{-1}} \mathcal{O}_\eta,$$

which takes values in the right hand side of the equality we want to prove. In fact, it is also surjective, since given an $f \in \mathcal{O}_\eta$ with $v_{x_i}(f; \mathcal{O}) \geq -n_i$ for each $x_i \in U$, $f \cdot s_D$ has nonnegative order at each point of U, so its domain of definition contains U and it extends to a section of $\mathcal{O}(D)(U)$. □

As a consequence we see that if we have Divisors $D = \sum n_x[x]$ and $D' = \sum n'_x[x]$ with $D \leq D'$, meaning that for each $x \in C$ we have $n_x \leq n'_x$, then $\mathcal{O}(D) \subseteq \mathcal{O}(D')$ as subsheaves of $K(C)$ as in Proposition 14.9.

DEFINITION 14.10. *Given any $f \in K^\times$ considered as a meromorphic section of $\mathcal{O}$ we can construct a divisor*

$$\operatorname{div}(f) := \sum_{x \in C} v_x(f; \mathcal{L})[x]$$

which is well-defined by Lemma 14.4.

THEOREM 14.11. *The following sequence*

$$0 \to k^\times \to K^\times \xrightarrow{\operatorname{div}} \operatorname{Div}(C) \xrightarrow{\mathcal{O}(-)} \operatorname{Pic}(C) \to 0$$

is exact.

PROOF. The injectivity of the first map is clear and the surjetivity of the last we have seen before (Corollary 13.2).
For exactness at $K^\times$ note that clearly $\operatorname{div}(\lambda) = 0$ for a constant function λ. Now assume that we have a function f with $\operatorname{div}(f) = 0$.Then it doesn't have poles, so it is a global function, hence constant.

Now to see that $\mathcal{O}(\operatorname{div}(f)) \cong \mathcal{O}$ we simply note that multiplication by f gives a map

$$\begin{aligned}\mathcal{O}(\mathrm{div(f)})(U) &= \{g \in K \mid v_{x_i}(g;\mathcal{O}) \geq -v_{x_i}(f;\mathcal{O}) \text{ for each } x_i \in U\} \\ &\xrightarrow{\cdot f} \{g \in K \mid v_{x_i}(g;\mathcal{O}) \geq 0 \text{ for each } x_i \in U\} = \mathcal{O}(U)\end{aligned}$$

with inverse given by multiplication with f^{-1}.
Finally assume that we have a divisor $D = \sum n_x[x]$ with an isomorphism $\mathcal{O}(D) \cong \mathcal{O}$. Under this isomorphism the global section $1 \in \mathcal{O}$ corresponds to a global section of $\mathcal{O}(D)$, i.e. an element $f \in K^\times$ with

$$v_x(f,\mathcal{O}) \geq -n_x$$

for all $x \in C$. In fact, by the fact that 1 does neither of poles nor zeros we see that we even have an equality

$$v_x(f,\mathcal{O}) = -n_x$$

so that $\operatorname{div}(f) = -D$. □

We now want to discuss some applications of Riemann-Roch. Abstractly, Riemann-Roch allows us to find functions with desired properties. We can use this to classify curves of small genus. First we make an observation similar to what we used when we related ampleness and quasi-projectiveness:

Lemma 14.12. *Let X be a scheme over* $\operatorname{Spec}(k)$ *and $\mathcal{L}$ a line bundle with sections $f_0, \dots, f_n$ such that:*

1. $\bigcup D(f_i) = X$,
2. *There exists a $m \in \mathbb{N}$ such that $\mathcal{L}^{\otimes k}(X)$ is generated by degree k monomials in the f_i for all $k \geq m$.*

Then the f_i determine a closed immersion $X \to \mathbb{P}^n_k$.

Proof. The sections f_i determine an epimorphism

$$\mathcal{O}^{n+1} \to \mathcal{L},$$

or dually a locally split monomorphism $\mathcal{L}^{-1} \to \mathcal{O}^{n+1}$, hence a morphism $f : X \to \mathbb{P}^n_k$. We have $f^*\mathcal{O}(1) \cong \mathcal{L}$ by construction, and $x_i \in \mathcal{O}(1)(\mathbb{P}^n_k)$ pulls back to f_i. In particular $f^{-1}(D(f_i)) = D(x_i)$. Furthermore, on each $D(f_j)$, the map

$$\mathcal{O}_{\mathbb{P}^n_k}(D(x_j)) = (f_*\mathcal{O}_{\mathbb{P}^n_k})(D(f_j)) \to \mathcal{L}(D(f_j))$$

takes $\frac{x_i}{x_j}$ to $\frac{f_i}{f_j}$ in

$$\mathcal{L}(D(f_j)) = \operatorname{colim}\left(\dots \to \mathcal{L}^k(X) \xrightarrow{f_j} \mathcal{L}^{k+1}(X) \to \dots\right).$$

By assumption, the $\frac{f_i}{f_j}$ generate the right hand side as algebra, and so $D(f_i) \to D(x_i)$ is a closed immersion. It follows that $X \to \mathbb{P}^n_k$ is a closed immersion. □

Theorem 14.13. *If C is a smooth projective curve over an algebraically closed field k, and C has genus 0, then $C \cong \mathbb{P}^1_k$.*

Proof. Pick a point $x \in C$, and consider the sheaf $\mathcal{O}(x)$. By Corollary 13.15, we see that $H^1(C;\mathcal{O}(x)) = 0$ and $H^0(C;\mathcal{O}(x))$ is 2-dimensional. We have that $H^0(C;\mathcal{O}(x))$ consists of meromorphic functions that only have a pole at x of pole order maximally 1. There is the function 1. There has to be another linearly

dependent function f as well, which has to actually have a pole at x, otherwise it would be globally defined and thus constant.
Now we have that

$$\mathcal{O}(x)^{\otimes d}(C) = \mathcal{O}(d[x])(C)$$

contains f^d with a pole of order d at x. Thus for any $g \in \mathcal{O}(d[x])(C)$ we have that some $g - \lambda f^d$ has a pole of a lower order, so that inductively $\mathcal{O}(d[x])(C)$ is generated by monomials in f. So by Lemma 14.12, they determine a closed immersion $C \to \mathbb{P}^1$. Since both have the same dimension and are reduced, this must be an isomorphism. □

THEOREM 14.14. *If C is a smooth projective curve of dimension 1 over an algebraically closed field k and C has genus 1, then C is isomorphic to a projective variety of the form*

$$V(zy^2 - x^3 - a_2xyz - a_3x^2z - a_4yz^2 - a_5xz^2 - a_6z^3) \subseteq \mathbb{P}^2_k.$$

if k has characteristic $\neq 2, 3$ then C is even isomorphic to a projective variety of the form

$$V(zy^2 - x^3 - axz^2 - bz^3) \subseteq \mathbb{P}^2_k.$$

PROOF. We pick a point $p \in C$, and study the line bundles $\mathcal{O}(d \cdot [p])$. Using Riemann–Roch (and Corollary 13.15) we get that

d	-2	-1	0	1	2	3	4	5	6
$\dim H^0(C; \mathcal{O}(d \cdot [p]))$	0	0	1	1	2	3	4	5	6
$\dim H^1(C; \mathcal{O}(d \cdot [p]))$	2	1	1	0	0	0	0	0	0

This tells us that $\mathcal{O}(p)(C)$ is generated by $1 \in K$, different from the genus 0 case. However, in $\mathcal{O}(2[p])(C)$, we have another section, say f which is not constant, hence has to have a pole or order 2 at p (it can't be of order 1 since then it would already lie in $\mathcal{O}(1[p])(C)$). Then $1, f$ form a basis of $\mathcal{O}(2[p])(C)$. In $\mathcal{O}(3[p])(C)$ we have $1, f$ need a further element $g \in K$. Again, K must have a pole of order 3 at p.
We now let $\mathcal{L} = \mathcal{O}(3[p])$, and write $z = 1$, $y = g$, $x = f$. We can now extend the table of sections

d	0	1	2	3	4	5	6	...
$\dim H^0(C; \mathcal{O}(d \cdot [p]))$	1	1	2	3	4	5	6	...
mero. functions	z		x	y	x^2	xy	x^3, y^2	...

We first see that every section of $\mathcal{L}^d = \mathcal{O}(3d[p])$ is a linear combination of monomials of x, y, z. By Lemma 14.12, this means that x, y, z determine a closed immersion into $\mathbb{P}^2_k$. To determine its image, we look for a relation between x, y, z. Moreover we learn that there is a relation of the form

$$a_0zy^2 + a_1x^3 + a_2xyz + a_3x^2z + a_4yz^2 + a_5xz^2 + a_6z^3 = 0.$$

for $a_i \in k$. Changing x, y by suitable units, we may assume $a_0 = -1$ and $a_1 = 1$. Thus the equation becomes the Weierstraßequation on an elliptic curve.
A substitution $y' = y + \lambda_1 x + \lambda_2 z$ absorbs the xyz, yz^2 terms into the zy^2 term, and then a substitution $x' = x + \mu z$ absorbs the x^2z term into the x^3 term. (Here we used the assumption on characteristic.) Finally, we have chosen x, y, z so that

$$zy^2 = x^3 + a_5xz^2 + a_6z^3.$$

Thus, our immersion $C \to \mathbb{P}^2_k$ maps C to $V(zy^2 - x^3 - a_5xz^2 - a_6z^3) \subseteq \mathbb{P}^2_k$. One easily checks this to be reduced, irreducible and of dimension 1, so $C \cong V(zy^2 - x^3 - a_5xz^2 - a_6z^3)$. $\square$